

THE KÄHLER-RICCI FLOW THROUGH SINGULARITIES ¹

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Abstract We prove the existence and uniqueness of the weak Kähler-Ricci flow on projective varieties with log terminal singularities. It is also shown that the weak Kähler-Ricci flow can be uniquely continued through divisorial contractions and flips if they exist. We then propose an analytic version of the Minimal Model Program with Ricci flow.

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1 Introduction

It has been the subject of intensive study over the last few decades to understand the existence of canonical Kähler metrics of Einstein type on a compact Kähler manifold, following Yau's solution to the Calabi conjecture (cf. [Y2], [A], [Y3], [T1], [T2]). The Ricci flow (cf. [Ha], [Ch]) provides a canonical deformation of Kähler metrics toward such canonical metrics. Cao [C] gives an alternative proof of the existence of Kähler-Einstein metrics on a compact Kähler manifold with numerically trivial or ample canonical bundle by the Kähler-Ricci flow. However, most projective manifolds do not have a numerically definite or trivial canonical bundle. It is a natural question to ask if there exist any well-defined canonical metrics on these manifolds or on varieties canonically associated to them. A projective variety is minimal if its canonical bundle is nef (numerically effective) and many results have been obtained on the Kähler-Ricci flow on minimal varieties. Tsuji [Ts] applies the Kähler-Ricci flow and proves the existence of a canonical singular Kähler-Einstein metric on a minimal projective manifold of general type. It is the first attempt to relate the Kähler-Ricci flow and canonical metrics to the Minimal Model Program. Since then, many interesting results have been achieved in this direction. The long time existence of the Kähler-Ricci flow on a minimal projective manifold with any initial Kähler metric is established in [TiZha]. The regularity problem of the canonical singular Kähler-Einstein metrics on minimal projective manifolds of general type is intensively studied in [EGZ1] and [Z1] independently. If the minimal projective manifold has positive Kodaira dimension and it is not of general type, it admits an Iitaka fibration over its canonical model. The authors define on the canonical model a new family of generalized Kähler-Einstein metrics twisted by a canonical form of Weil-Petersson type from the fibration structure ([SoT1], [SoT2]). It is also proved by the authors that the normalized Kähler-Ricci flow converges to such a canonical metric if the canonical bundle is semi-ample ([SoT1], [SoT2]).

Let X be an n -dimensional projective manifold. We consider the following unnormalized Kähler-Ricci flow starting with a Kähler metric $\omega_0 \in H^{1,1}(X, \mathbf{R}) \cap H^2(X, \mathbf{Z})$.

$$\begin{cases} \frac{\partial}{\partial t} \omega = -Ric(\omega) \\ \omega|_{t=0} = \omega_0. \end{cases} \quad (1.1)$$

The unnormalized Kähler-Ricci flow (1.1) has long time existence if X is a minimal model, i.e., the canonical bundle K_X is nef ([TiZha]).

If K_X is not nef, the unnormalized Kähler-Ricci flow (1.1) must become singular at certain time $T_0 > 0$. At time T_0 , either the flow develops singularities on a subvariety of X or X admits a Fano fibration and the flow is expected to collapse along the fibres. For the first case, the subvariety where the singularities appear is exactly where K_X is negative. The flow then might perform an analytic/geometric surgery equivalent to an algebraic surgery such as a divisorial contraction or a flip, and replace X by a new projective variety X' . Hopefully, the flow can be continued on X' , which usually has mild singularities. The main goal of the paper is to define the Kähler-Ricci flow on singular varieties such as X' and to construct analytic surgeries for the Kähler-Ricci flow. If the second case occurs and the flow (1.1) collapses onto a new projective variety X'' , we expect the flow can be continued on the base X'' . Heuristically, we can repeat the above procedures until either the flow exists for all time or it collapses to a point. If the flow exists for all time, it should converge to a generalized Kähler-Einstein metric on its canonical model or a Ricci-flat metric on its minimal model after normalization if we assume the abundance conjecture. Eventually, we arrive at the final case when X is Fano and

it is conjectured by the second named author that the unnormalized Kähler-Ricci flow becomes extinct in finite time after surgery if and only if it is birationally equivalent to a Fano variety [T3]. The conjecture is proved by the first named author for smooth solutions of the flow [So].

In general, the varieties obtained from divisorial contractions and flips have mild singularities. Also we expect that the analytic surgeries performed by the Kähler-Ricci flow coincide with the algebraic surgeries as divisorial contractions and flips. Therefore we can not avoid singularities if the Kähler-Ricci flow can be indeed continued through surgeries. We then must define the Kähler-Ricci flow on projective varieties with singularities.

We confine ourselves in the category of singularities considered in the Minimal Model Program because such singularities are rather mild and they do not get worse after divisorial contractions or flips are performed. The precise definition is given in Section 2.3 for a \mathbf{Q} -factorial projective variety with log terminal singularities. Roughly speaking, let X be such a projective normal variety and $\pi : \tilde{X} \rightarrow X$ be a resolution of singularity, then the pullback of any smooth volume form on X is integrable on the nonsingular model \tilde{X} .

Our first theorem proves the existence and uniqueness for the Kähler-Ricci flow on projective varieties with log terminal singularities. Furthermore, it establishes a smoothing property for the Kähler-Ricci flow if the initial data is not smooth.

Theorem A.1 *Let X be a \mathbf{Q} -factorial projective variety with log terminal singularities and H be an ample \mathbf{Q} -divisor on X . Let*

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}.$$

If $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$, then there exists a unique solution ω of the unnormalized weak Kähler-Ricci flow (1.1) starting with ω_0 for $t \in [0, T_0]$.

Furthermore, if Ω is a smooth volume form on X , then for any $T \in (0, T_0)$, there exists $C > 0$ such that on $[0, T] \times X$,

$$e^{-\frac{C}{t}}\Omega \leq \omega^n \leq e^{\frac{C}{t}}\Omega. \quad (1.2)$$

The definitions are given in Section 4.1 for $\mathcal{K}_{H,p}(X)$ (Definition 4.2) and the weak Kähler-Ricci flow (Definition 4.3). Theorem A.1 shows that the Kähler-Ricci flow can start with a Kähler current which admits bounded local potential and an L^p Monge-Ampère mass for some $p > 1$. It gives the short time existence for the weak Kähler-Ricci flow. Furthermore, it smoothes out the initial current in the sense that the flow becomes smooth on the nonsingular part of X once $t > 0$ and the evolving metrics always admit bounded local potentials for any $t \in (0, T_0)$. In particular, if X is nonsingular, the flow becomes the usual Kähler-Ricci flow with smooth solutions on $(0, T_0) \times X$. In particular, T_0 is exactly the first singular time for the unnormalized weak Kähler-Ricci flow.

It is not clear how to define metrics on a singular variety X with reasonable regularity and curvature conditions in general. One natural choice is the restriction of the Fubini-Study metric ω_{FS} for some projective embedding of X , if X is normal and projective. It is indeed a smooth metric on X , however, even the scalar curvature of ω_{FS} might blow up near the singularities of X . More seriously, $(\omega_{FS})^n$ might not be a smooth volume form on X in general, although it is a smooth non-negative (n, n) -form on X (see Section 4.3 for more detailed discussions). Theorem A.1 shows that the volume form of the corresponding solutions of the weak Kähler-Ricci flow becomes equivalent to a smooth volume form immediately for $t > 0$. We speculate that the weak Kähler-Ricci flow produces metrics on X with reasonably good geometric conditions. For

example, given a normal projective orbifold X embedded in some projective space \mathbf{CP}^N , the Fubini-Study metric ω_{FS} is in general not a smooth orbifold Kähler metric on X . If we start the Kähler-Ricci flow with ω_{FS} , the evolving metrics immediately become smooth orbifold Kähler metrics on X .

We also make a remark that the assumption of \mathbf{Q} -factoriality can be weakened. In fact, the theorems still hold with slight modification if both the initial divisor H and K_X are \mathbf{Q} -Cartier.

The following theorem shows that the Kähler-Ricci flow can be defined on smooth projective varieties if the initial class is not Kähler.

Theorem A.2 *Let X be a non-singular projective variety and H be a big and semi-ample \mathbf{Q} -divisor on X . Suppose that*

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\} > 0.$$

If $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$, then there exists a unique solution ω of the unnormalized weak Kähler-Ricci flow (1.1) for $t \in [0, T_0]$.

Furthermore, for any $t \in (0, T_0)$, there exists $C(t) > 0$ such that

$$\|S(\omega(t, \cdot))\|_{L^\infty(X)} \leq C(t), \quad (1.3)$$

where $S(\omega(t, \cdot))$ is the scalar curvature of $\omega(t, \cdot)$.

The scalar curvature S is defined on a Zariski open set of X away from the exceptional locus of H . Theorem A.2 shows that for each $t \in (0, T_0)$, the scalar curvature is uniformly bounded.

Theorem A.2 immediately implies the following corollary. It turns out that at each $t \in (0, T_0)$, the evolving metric has bounded scalar curvature on a projective variety which admits a crepant resolution.

Corollary A.3 *Let X be a \mathbf{Q} -factorial projective variety with crepant singularities, H be an ample \mathbf{Q} -divisor on X and*

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}.$$

If $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$, then there exists a unique solution ω of the unnormalized weak Kähler-Ricci flow (1.1) for $t \in [0, T_0]$.

Furthermore, for any $t \in (0, T_0)$, there exists $C(t) > 0$ such that

$$\|S(\omega(t, \cdot))\|_{L^\infty(X)} \leq C(t), \quad (1.4)$$

where $S(\omega(t, \cdot))$ is the scalar curvature of $\omega(t, \cdot)$.

From now on, we always assume X is a \mathbf{Q} -factorial projective variety with log terminal singularities and H be an ample \mathbf{Q} -divisor on X . Let

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}$$

be the first singular time the unnormalized weak Kähler-Ricci flow (1.1) for $t \in [0, T_0]$ starting with $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$.

Theorem A.1 gives the short time existence of the weak unnormalized Kähler-Ricci flow and the first singular time T_0 is exactly when the Kähler class of the evolving metrics stops being nef. If $T_0 < \infty$ and the limiting Kähler class is big, there is a contraction morphism $\pi : X \rightarrow Y$

uniquely associated to the limiting divisor $H + T_0 K_X$. Let $\overline{NE}(X)$ be the closure of the convex cone that consists of the classes of effective curves on X . If the morphism π contracts exactly one extremal ray of $\overline{NE}(X)$, the recent result of [BCHM] and [HM] shows that either π contracts a divisor or there exists a unique flip associated to π (see Definition 5.4 for a flip).

Since the weak unnormalized Kähler-Ricci flow cannot be continued on X at the singular time T_0 , we have to replace X by another variety X' and continue the flow on X' . Our next main result is to relate the finite time singularities of the unnormalized Kähler-Ricci flow (1.1) to divisorial contractions and flips in the Minimal Model Program.

Theorem B.1 *Let ω be the unique solution of the unnormalized weak Kähler-Ricci flow (1.1) for $t \in [0, T_0)$ starting with $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$. Suppose that $H + T_0 K_X$ is big and the morphism $\pi : X \rightarrow Y$ induced by the semi-ample divisor $H_{T_0} = H + T_0 K_X$ contracts exactly one extremal ray of $\overline{NE}(X)$.*

1. *If π is a divisorial contraction, then there exists $\omega_{Y,0} \in \mathcal{K}_{H_Y,p'}(Y) \cap C^\infty(Y_{reg} \setminus \pi(Exc(\pi)))$ for some $p' > 1$ such that, $\omega(t, \cdot)$ converges to $\pi^* \omega_{Y,0}$ in $C^\infty(X_{reg} \setminus Exc(\pi))$ -topology as $t \rightarrow T_0$, where $H_Y = \pi_* H_{T_0}$.*

Furthermore, the unnormalized weak Kähler-Ricci flow (1.1) can be continued on Y with the initial Kähler current $\omega_{Y,0}$.

2. *If π is a small contraction and there exists a flip*

$$\begin{array}{ccc} X & \xrightarrow{\pi^{-1}} & X^+ \\ \searrow \pi & & \swarrow \pi^+ \\ Y & & \end{array} , \quad (1.5)$$

then there exists $\omega_{X^+,0} \in \mathcal{K}_{H_{X^+},p'}(X^+)$ for some $p' > 1$, such that $\omega(t, \cdot)$ converges to $(\pi^{-1})^ \omega_{X^+,0}$ in $C^\infty(X_{reg} \setminus Exc(\pi))$ -topology, where H_{X^+} is the strict transformation of H_{T_0} by π .*

Furthermore, $\omega_{X^+,0}$ is smooth outside the singularities of X^+ and where the flip is performed, and the unnormalized weak Kähler-Ricci flow (1.1) can be continued on X^+ with the initial Kähler current $\omega_{X^+,0}$.

Here $Exc(\pi)$ denotes the exceptional locus of the morphism π and X_{reg} denotes the non-singular part of X . In summary, we have the following corollary. $(\pi^{-1})^* \omega_{X^+,0}$ is defined by pulling back the local potentials of $\omega_{X^+,0}$. It is well-defined because the local potential of $\omega_{X^+,0}$ can be chosen to be constant along each connected fibre of π^+ and in particular, $(\pi^{-1})^* \omega_{X^+,0} \in \mathcal{K}_{H_{T_0},p}(X)$.

Corollary B.2 *The unnormalized Kähler-Ricci flow can be continued through divisorial contractions and flips.*

The Minimal Model Program is successful in dimension three by Mori's work and the recent works have (c.f. [BCHM], [Si]) led to proving the finite generation of canonical rings. The deformation of the Kähler classes along the unnormalized Kähler-Ricci flow is in line with the Minimal Model Program with Scaling (MMP with scaling) proposed in [BCHM]. It is also

proved in [BCHM] that MMP with scaling terminates after finitely many divisorial contractions and flips if the variety X is of general type.

A good initial divisor H means that there are finitely many singular times and the contraction morphism at each singular time only contracts one extremal ray if the unnormalized Kähler-Ricci flow (1.1) starts with H . We refer the readers to Section 5.1 for the precise definition (Definition 5.3). In particular, good initial divisors always exist if $\dim X = 2$ and $\text{kod}(X) \geq 0$, then the normalized Kähler-Ricci flow with a good initial divisor converges to the canonical model or the minimal model of X coupled with a generalized Kähler-Einstein metric. It is possible that a general ample \mathbf{Q} -divisor H on X is a good initial divisor since MMP with scaling terminates for X .

Theorem C.1 *Let X be a projective \mathbf{Q} -factorial variety of general type with log terminal singularities. If H is a good initial divisor on X , then the normalized weak Kähler-Ricci flow*

$$\frac{\partial \omega}{\partial t} = -Ric(\omega) - \omega \quad (1.6)$$

starting with any initial Kähler current in $\mathcal{K}_{H,p}(X)$ for some $p > 1$ exists for $t \in [0, \infty)$ and it replaces X by its minimal model X_{\min} after finitely many surgeries. Furthermore, the normalized Kähler-Ricci flow converges in distribution to the unique Kähler-Einstein metric ω_{KE} on its canonical model X_{can} .

Theorem C.1 gives the general philosophy of the analytic Minimal Model Program with Ricci Flow. The Kähler-Ricci flow deforms a given projective variety X to its minimal model X_{\min} in finite time after finitely many metric surgeries. Then X_{\min} is deformed to the canonical model X_{can} coupled with a generalized Kähler-Einstein metric by the flow after normalization. We also remark that the flow converges in the sense of distribution globally and in the C^∞ -topology away from the singularities of X_{\min} and the exceptional locus of the pluricanonical system. Certainly, it is desired that the convergence should be in the sense of Gromov-Hausdorff. We also remark that when X is a nonsingular minimal model of general type, the convergence of the normalized Kähler-Ricci flow is proved in [Ts] and [TiZha].

The organization of the paper is the following. In Section 2, we set up the basic notations for degenerate complex Monge-Ampère equations and algebraic singularities in the minimal model theory. In Section 3, we solve a special family of degenerate parabolic Monge-Ampère equations on projective manifolds. In Section 4, we apply the results in Section 3 to prove Theorem A.1, Theorem A.2 and Corollary A.3 for the short time existence of the weak Kähler-Ricci flow. In Section 5, Theorem B.1 and Corollary B.2 are proved for the weak Kähler-Ricci flow through singularities. We also prove Theorem C.1 for long time existence and convergence. Finally in Section 6, we propose an analytic Minimal Model Program with Ricci Flow.

2 Preliminaries

2.1 Kodaira dimension and canonical measures

Let X be an n -dimensional compact complex projective manifold and $L \rightarrow X$ a holomorphic line bundle over X . Let $N(L)$ be the semi-group defined by

$$N(L) = \{m \in \mathbf{N} \mid H^0(X, L^m) \neq 0\}.$$

Given any $m \in N(L)$, the linear system $|L^m| = \mathbf{P}H^0(X, L^m)$ induces a rational map Φ_m

$$\Phi_m : X \dashrightarrow \mathbf{CP}^{d_m}$$

by any basis $\{\sigma_{m,0}, \sigma_{m,1}, \dots, \sigma_{m,d_m}\}$ of $H^0(X, L^m)$, where

$$\Phi_m(z) = [\sigma_{m,0}, \sigma_{m,1}, \dots, \sigma_{m,d_m}(z)],$$

and $d_m + 1 = \dim H^0(X, L^m)$. Let $Y_m = \overline{\Phi_m(X)} \subset \mathbf{CP}^{d_m}$ be the closure of the image of Φ_m .

Definition 2.1 *The Iitaka dimension of L is defined to be*

$$\kappa(X, L) = \max_{m \in N(L)} \{\dim Y_m\}$$

if $N(L) \neq \emptyset$, and $\kappa(X, L) = -\infty$ if $N(L) = \emptyset$.

Definition 2.2 *Let X be a projective manifold and K_X be the canonical line bundle over X . Then the Kodaira dimension $\text{kod}(X)$ of X is defined to be*

$$\text{kod}(X) = \kappa(X, K_X).$$

The Kodaira dimension is a birational invariant of a projective variety and the Kodaira dimension of a singular variety is equal to that of its smooth model.

Definition 2.3 *Let $L \rightarrow X$ be a holomorphic line bundle over a compact projective manifold X . L is called nef if $L \cdot C \geq 0$ for any curve C on X and L is called semi-ample if L^m is globally generated for some $m > 0$.*

For any $m \in \mathbf{N}$ such that L^m is globally generated, the linear system $|L^m|$ induces a holomorphic map Φ_m

$$\Phi_m : X \rightarrow \mathbf{CP}^{d_m}$$

by any basis of $H^0(X, L^m)$. Let $Y_m = \Phi_m(X)$ and so Φ_m can be considered as

$$\Phi_m : X \rightarrow Y_m.$$

The following theorem is well-known (cf. [La, U]).

Theorem 2.1 *Let $L \rightarrow X$ be a semi-ample line bundle over an algebraic manifold X . Then there is an algebraic fibre space*

$$\Phi_\infty : X \rightarrow Y$$

such that for any sufficiently large integer m with L^m being globally generated,

$$Y_m = Y \quad \text{and} \quad \Phi_m = \Phi_\infty,$$

where Y is a normal projective variety. Furthermore, there exists an ample line bundle A on Y such that $L^m = (\Phi_\infty)^* A$.

If L is semi-ample, the graded ring $R(X, L) = \oplus_{m \geq 0} H^0(X, L^m)$ is finitely generated and so it is the coordinate ring of Y .

Let X be an n -dimensional projective manifold. It is recently proved in [BCHM] and [Si] independently that the canonical ring $R(X, K_X)$ is finitely generated if X is of general type. Then the canonical ring induces a rational map from X to its unique canonical model X_{can} . The following theorem is proved in [EGZ1] when X is of general type (also see [Ts], [TiZha] for minimal models of general type) and in [SoT2] when X admits an Iitaka fibration over X_{can} .

Theorem 2.2 *Let X be an n -dimensional projective manifold with $R(X, K_X)$ being finitely generated.*

1. $\text{kod}(X) = n$, then there exists a unique Kähler current $\omega_{KE} \in [K_{X_{can}}]$ with bounded local potential satisfying the Kähler-Einstein equation

$$\text{Ric}(\omega_{KE}) = -\omega_{KE}. \quad (2.1)$$

2. $0 < \text{kod}(X) < n$, X admits a rational fibration over X_{can} whose general fibre has Kodaira dimension 0. There exists a unique Kähler current $\omega_{can} \in [K_{X_{can}} + L_{X/X_{can}}]$ such that

$$\text{Ric}(\omega_{can}) = -\omega_{can} + \omega_{WP}, \quad (2.2)$$

where $L_{X/X_{can}}$ is relative dualizing sheaf and ω_{WP} is a canonical current of Weil-Peterson type induced from the Calabi-Yau fibration.

The closed current ω_{WP} is exactly the pullback of the Weil-Peterson metric on the moduli space of Calabi-Yau varieties associated to the fibres if X is a smooth minimal model and K_X is semi-ample. When X is not minimal, the general fibre is not necessary a Calabi-Yau variety. One can still define ω_{WP} as a L^2 -metric on the deformation space for varieties of 0 Kodaira dimension. We refer the readers to the precise definition in [SoT2].

If $\text{kod}(X) = 0$, then so there exists a holomorphic volume form $\Omega = (\eta \otimes \bar{\eta})^{1/m}$ for some holomorphic section $\eta \in H^0(X, mK_X)$. It is proved in [SoT2] that for any ample divisor H on X , there exists a Kähler current $\omega_{CY} \in H$ with bounded local potential such that

$$(\omega_{CY})^n = c\Omega \quad (2.3)$$

for some positive constant $c > 0$. Therefore $\text{Ric}(\omega_{CY}) = 0$ outside the stable base locus of the pluricanonical system of X . The existence of singular Ricci-flat Kähler metrics is proved in [EGZ1] on singular Calabi-Yau varieties.

Such metrics are the unique canonical metrics on projective varieties of non-negative Kodaira dimension and the generalized Kähler-Einstein equations can be viewed as an analytic version of the adjunction formula. They are candidates for the limiting metrics of the Kähler-Ricci flow.

2.2 Complex Monge-Ampère equations

In this section, we review some of the important results in degenerate complex Monge-Ampère equations developed by Kolodziej [Ko1] and many others ([Z1], [EGZ1], [DP], [EGZ2]). We start with some basic notations.

Definition 2.4 *Let X be an n -dimensional Kähler manifold and ω be a closed semi-positive $(1, 1)$ -current on X .*

1. ω is Kähler if it is positive.
2. ω is called big if $[\omega]^n = \int_X \omega^n > 0$.
3. ω is called a Kähler current if it is big.

If ω is a Kähler current with bounded local potential on X , the corresponding volume current ω^n is uniquely well-defined by the standard pluripotential theory.

Definition 2.5 *Let ω be a Kähler current with bounded local potential on X . A quasi-plurisubharmonic function associated to ω is an upper semi-continuous function $\varphi : X \rightarrow [-\infty, \infty)$ such that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi \geq 0$. We denote by $PSH(X, \omega)$ the set of all quasi-plurisubharmonic functions associated to ω on X .*

In [Ko1], Kolodziej proves the fundamental theorem on the existence of continuous solutions to the Monge-Ampère equation $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = F\omega^n$, where ω is a Kähler form and $F \in L^p(X, \omega^n)$ for some $p > 1$ is non-negative. Its generalization was independently carried out in [Zh] and [EyGuZe1]. They prove that there is a bounded solution when ω is semi-positive and big. A detailed proof for the continuity of the solution was given in [DZ] (also see [Zh] for an earlier sketch of proof). These generalizations are summarized in the following.

Theorem 2.3 *Let X be an n -dimensional Kähler manifold and let ω be a Kähler current with bounded local potential. Then there exists a unique solution $\varphi \in PSH(X, \omega) \cap L^\infty(X)$ solving the following Monge-Ampère equation*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = F\Omega,$$

where $\Omega > 0$ is a smooth volume form on X , $F \in L^p(X, \Omega)$ for some $p > 1$ and $\int_X F\Omega = \int_X \omega^n$.

In [Ko2], Kolodziej proves the stability result for solutions of the complex Monge-Ampere equations for Kähler classes. It is later improved by Dinew and Zhang [DZ] (also see [DP] more general cases) for big and semi-ample classes. The following is a version of their result.

Theorem 2.4 *Let X be an n -dimensional compact Kähler manifold. Suppose $L \rightarrow X$ is a semi-ample line bundle and $\omega \in c_1(L)$ is a smooth Kähler current. Let Ω be a smooth volume form on X . For any non-negative functions f and $g \in L^p(X, \Omega)$ for some $p > 1$ with $\int_X f\Omega = \int_X g\Omega$, there exist φ and $\psi \in PSH(X, \omega) \cap L^\infty(X)$ solving*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = f\Omega, \quad (\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = g\Omega$$

with

$$\max_X(\varphi - \psi) = \max_X(\psi - \varphi).$$

Then for any $\epsilon > 0$, there exists $C > 0$ depending on ϵ and p , $\|f\|_{L^p(X, \Omega)}$ and $\|g\|_{L^p(X, \Omega)}$ such that

$$\|\varphi - \psi\|_{L^\infty(X)} \leq C\|f - g\|_{L^1(X, \Omega)}^{\frac{1}{n+3+\epsilon}}. \quad (2.4)$$

Theorem 2.4 can be generalized for the case where the right hand side of the Monge-Ampere equations contains terms such e^φ by Kolodziej's argument in [Ko2]. Theorem 2.4 also holds uniformly for certain family of ω , such as $\omega + \epsilon\chi$ with a fixed Kähler metric χ and $\epsilon \in [0, 1]$. Also the sharper exponents are obtained in [DZ] and [DP].

2.3 Singularities

We will have to study the behavior of the Kähler-Ricci flow on normal projective varieties with singularities because the original smooth manifold might be replaced by varieties with mild singularities through surgery along the flow.

The pluripotential theory on normal varieties has been extensively studied (cf [FN]). Let X be a normal variety. A function f on X is continuous (or smooth) if f can be extended to a continuous (or smooth) function in a local embedding from X to \mathbf{C}^N . A plurisubharmonic function is an upper semi-continuous function $\varphi : V \rightarrow [-\infty, \infty)$ which locally extends to a plurisubharmonic function in a local embedding from X to \mathbf{C}^N . By the work of [?, ?] any bounded plurisubharmonic function on X_{reg} , the nonsingular part of X , can be uniquely extended to a plurisubharmonic function on X . Let X be a normal projective variety and ω be a semi-positive closed $(1, 1)$ -current on X . We let $PSH(X, \omega)$ be the set of all upper semi-continuous functions $\varphi : X \rightarrow [-\infty, \infty)$ such that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi \geq 0$.

In this paper, we confine our discussions to projective varieties with mild singularities which are allowed in the Minimal Model Program in algebraic geometry.

Definition 2.6 [KMM] *Let X be a normal projective variety such that K_X is a \mathbf{Q} -Cartier divisor. Let $\pi : \tilde{X} \rightarrow X$ be a resolution and $\{E_i\}_{i=1}^p$ the irreducible components of the exceptional locus $Exc(\pi)$ of π . There there exists a unique collection $a_i \in \mathbf{Q}$ such that*

$$K_{\tilde{X}} = \pi^*K_X + \sum_{i=1}^p a_i E_i.$$

Then X is said to have

- *terminal singularities if $a_i > 0$, for all i .*
- *canonical singularities if $a_i \geq 0$, for all i .*
- *log terminal singularities if $a_i > -1$, for all i .*
- *log canonical singularities if $a_i \geq -1$, for all i .*

Terminal, canonical and log terminal singularities are always rational, while log canonical singularities are not necessarily rational. We can always assume that the resolution π is good enough such that the exceptional locus is a simple normal crossing divisor.

Definition 2.7 *A variety X is \mathbf{Q} -factorial if any \mathbf{Q} -Weil divisor on X is \mathbf{Q} -Cartier.*

Kodaira's lemma states that for any big and nef line divisor H on X , there always exists an effective divisor E such that $H - \epsilon E$ is ample for any sufficiently small $\epsilon > 0$. Let $\pi : \tilde{X} \rightarrow X$ be a birational morphism between two projective varieties and $Exc(\pi)$ be the exceptional locus of π , where π is not isomorphic. The following proposition is a special case of Kodaira's lemma and the support of E exactly coincides with $Exc(\pi)$ (see [D]).

Proposition 2.1 *If X is normal and \mathbf{Q} -factorial, then for any ample \mathbf{Q} -divisor H on X , there exists an effective divisor E on \tilde{X} whose support is $Exc(\pi)$ and $\pi^*H - \epsilon E$ is ample for sufficiently small $\epsilon > 0$.*

It is also well-known that \mathbf{Q} -factoriality is preserved after divisorial contractions and flips in the Minimal Model Program. \mathbf{Q} -factoriality is a necessary condition in our discussion because we need the canonical divisor to be a Cartier \mathbf{Q} -divisor in order to define a volume form appropriately.

3 Monge-Ampère flows

3.1 Monge-Ampère flows with rough initial data

In this section, we will prove the smoothing property of the Kähler-Ricci flow with rough initial data. We will assume that X is an n -dimensional Kähler manifold.

Definition 3.1 Suppose ω is a Kähler form and Ω is a smooth volume form on X . Then we define for $p \in (0, \infty]$,

$$PSH_p(X, \omega, \Omega) = \{\varphi \in PSH(X, \omega) \cap L^\infty(X) \mid \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} \in L^p(X)\}.$$

Note that $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n$ is a well-defined Monge-Ampère mass for bounded ω -psh function φ . In Definition 3.1, the Monge-Ampère mass $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n$ must be absolutely continuous with respect to Ω in order to define $\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} \in L^p(X)$. Suppose $\varphi_0 \in PSH_p(X, \omega_0, \Omega)$ for some $p > 1$. Let

$$F = \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0)^n}{\Omega} \in L^p(X).$$

By Kolodziej's result [Ko1],

$$\varphi_0 \in C^0(X).$$

The following proposition shows that any element in $PSH_p(X, \omega, \Omega)$ for $p > 1$ can be uniformly approximated by smooth quasi-plurisubharmonic functions.

Proposition 3.1 There exist a sequence $\{\varphi_{0,j}\}_{j=1}^\infty \subset PSH(X, \omega_0) \cap C^\infty(X)$ such that

$$\lim_{j \rightarrow \infty} \|\varphi_{0,j} - \varphi_0\|_{L^\infty(X)} = 0. \quad (3.1)$$

Proof Recall that $C^\infty(X)$ is dense in $L^p(X)$. Therefore there exists a sequence of positive functions $\{F_j\} \in C^\infty(X)$ such that $\int_X F_j \Omega = \int_X F \Omega$ and

$$\lim_{j \rightarrow \infty} \|F_j - F\|_{L^p(X)} = 0.$$

We then consider the solutions of the following Monge-Ampère equations

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{0,j})^n = F_j \Omega. \quad (3.2)$$

Since $F_j \in C^\infty(X)$ and $F_j > 0$, $\varphi_{0,j} \in PSH(X, \omega_0) \cap C^\infty(X)$. Without loss of generality, we can assume

$$\sup_X (\varphi_0 - \varphi_{0,j}) = \sup_X (\varphi_{0,j} - \varphi_0).$$

By the stability theorem of Kolodziej [Ko2] (Theorem 2.4), we have

$$\|\varphi_{0,j} - \varphi_0\|_{L^\infty(X)} \leq C \|F_j - F\|_{L^1(X)}^{\frac{1}{n+4}}$$

where C only depends on $\|F_j\|_{L^p(X)}$ and $\|F\|_{L^p(X)}$. The proposition follows easily. \square

Now we have a sequence of smooth Kähler forms

$$\omega_{0,j} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{0,j}.$$

Let $\chi = \sqrt{-1}\partial\bar{\partial}\log\Omega \in [K_X]$ and $\omega_t = \omega_0 + t\chi$. By simple calculation, one can show that the unnormalized Kähler-Ricci flow with the initial Kähler metric $\omega_{0,j}$ is equivalent to the following Monge-Ampère flow

$$\begin{cases} \frac{\partial\varphi_j}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_j)^n}{\Omega} \\ \varphi_j(0, \cdot) = \varphi_{0,j}. \end{cases} \quad (3.3)$$

We define

$$T_0 = \sup\{t \geq 0 \mid [\omega_0] + t[K_X] \text{ is nef}\}$$

to be the first time when the Kähler class stops being positive along the Kähler-Ricci flow. It is well-known that $T_0 > 0$ and by the result of [TiZha], the Monge-Ampère flow exists for $[0, T_0]$.

The following lemma shows that the Monge-Ampère flows starting with $\varphi_{0,j}$ approximate the same flow starting with φ_0 .

Lemma 3.1 *For any $0 < T < T_0$, there exists $C > 0$ such that for $t \in [0, T]$,*

$$\|\varphi_j\|_{L^\infty(X)} \leq C. \quad (3.4)$$

Furthermore, $\{\varphi_j\}$ is a Cauchy sequence in $L^\infty([0, T] \times X)$, i.e.,

$$\lim_{j,k \rightarrow \infty} \|\varphi_j - \varphi_k\|_{L^\infty([0, T] \times X)} = 0. \quad (3.5)$$

Proof Applying the maximum principle to φ_j , we can show that there exists $C > 0$ such that

$$\sup_{[0, T] \times X} |\varphi_j| \leq T \sup_X \left| \log \frac{\omega_t^n}{\Omega} \right| + \sup_X |\varphi_{0,j}| \leq C.$$

Let $\psi_{j,k} = \varphi_j - \varphi_k$. Then $\psi_{j,k}$ satisfies the following equation

$$\begin{cases} \frac{\partial\psi_{j,k}}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_k + \sqrt{-1}\partial\bar{\partial}\psi_{j,k})^n}{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_k)^n} \\ \psi_{j,k}(0, \cdot) = \varphi_{0,j} - \varphi_{0,k}. \end{cases} \quad (3.6)$$

By the maximum principle,

$$\sup_{[0, T] \times X} |\varphi_j - \varphi_k| = \sup_{[0, T] \times X} |\psi_{j,k}| \leq \sup_X |\varphi_{0,j} - \varphi_{0,k}|.$$

Then

$$\lim_{j,k \rightarrow \infty} \|\varphi_j - \varphi_k\|_{L^\infty([0, T] \times X)} \leq \lim_{j,k \rightarrow \infty} \|\varphi_{0,j} - \varphi_{0,k}\|_{L^\infty(X)} = 0.$$

□

We also can bound the volume form along the Monge-Ampère flow, even though the initial volume form is only in $L^p(X)$.

Lemma 3.2 For any $0 < T < T_0$, there exists $C > 0$, such that for $t \in [0, T]$.

$$\frac{t^n}{C} \leq \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_j)^n}{\Omega} \leq e^{\frac{C}{t}}. \quad (3.7)$$

Proof Let Δ_j be the Laplacian operator associated to the Kähler form $\omega_j = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_j$. Straightforward calculations show that $(\frac{\partial}{\partial t} - \Delta_j)\dot{\varphi}_j = \text{tr}_{\omega_j}(\chi)$.

Let $H^+ = t\dot{\varphi}_j - \varphi_j$. Then $H^+(0, \cdot) = -\varphi_j$ is uniformly bounded and

$$(\frac{\partial}{\partial t} - \Delta_j)H^+ = -\text{tr}_{\omega_j}(\omega_t - t\chi) + n = -\text{tr}_{\omega_j}(\omega_0) + n \leq n.$$

By the maximum principle, H^+ is uniformly bounded from above for $t \in [0, T]$.

Let $H^- = \dot{\varphi}_j + A\varphi_j - n \log t$. Then $H^-(t, \cdot)$ tends to ∞ uniformly as $t \rightarrow 0^+$ and there exist constants C_1, C_2 and $C_3 > 0$ such that

$$\begin{aligned} (\frac{\partial}{\partial t} - \Delta_j)H^- &= \text{tr}_{\omega_j}(A\omega_t + \chi) + A\dot{\varphi}_j - \frac{n}{t} - An \\ &\geq C_1\left(\frac{\omega_0^n}{\omega_j^n}\right)^{\frac{1}{n}} + A \log \frac{\omega_j^n}{\Omega} - \frac{n}{t} - An \\ &\geq C_2\left(\frac{\omega_0^n}{\omega_j^n}\right)^{\frac{1}{n}} - \frac{C_3}{t}, \end{aligned}$$

if A is chosen sufficiently large such that $A\omega_t + \chi \geq \omega_0$ for $t \in [0, T]$. Then at the minimal point of H^- , the maximum principle gives

$$\omega_j^n \geq C_4 t^n \Omega.$$

It easily follows that H^- is uniformly bounded from below for $t \in [0, T]$.

Since φ_j is uniformly bounded for $t \in [0, T]$, the lemma is proved. \square

The following smoothing lemma shows that the approximating metrics become uniformly bounded immediately along the Monge-Ampère flow.

Lemma 3.3 For any $0 < T < T_0$, there exists $C > 0$ such that for $t \in (0, T]$,

$$\text{tr}_{\omega_0}(\omega_j) \leq e^{\frac{C}{t}}. \quad (3.8)$$

Proof This is a parabolic Schwarz lemma similar to [Y1], [LiYa]. Straightforward computation from [SoT1] shows that for any $t \in [0, T]$, there exist uniform constants C_1 and $C_2 > 0$ such that

$$(\frac{\partial}{\partial t} - \Delta_j) \log \text{tr}_{\omega_0}(\omega_j) \leq C_1 \text{tr}_{\omega_j}(\omega_0) + C_2.$$

Let $H = t \log \text{tr}_{\omega_0}(\omega_j) - A\varphi_j$. Then if A is sufficiently large, there exist uniform constants $C_3, C_4, \dots, C_{10} > 0$ such that

$$\begin{aligned}
(\frac{\partial}{\partial t} - \Delta_j)H &\leq -tr_{\omega_j}(A\omega_t - C_1 t\omega_0) - A\varphi_j + \log tr_{\omega_0}(\omega_j) + C_3 \\
&\leq -C_4 tr_{\omega_j}(\omega_0) + C_5 \log tr_{\omega_j}(\omega_0) - C_6 \log \frac{\omega_j^n}{\omega_0^n} + C_7 \\
&\leq -C_8 \left(\frac{\omega_0^n}{\omega_j^n} \right)^{\frac{1}{n-1}} (tr_{\omega_0}(\omega_j))^{\frac{1}{n-1}} - C_9 \log t + C_{10}.
\end{aligned}$$

Suppose $\max_{[0,T] \times X} H = H(t_0, z_0)$. Since $H(0, \cdot) = -\infty$, $t_0 > 0$. Then by the maximum principle, at (t_0, z_0) ,

$$\log tr_{\omega_0}(\omega_j) \leq \log \left((\log \frac{1}{t})^{n-1} \left(\frac{\omega_j^n}{\omega_0^n} \right) \right) + C_{11} \leq C_{12} \frac{1}{t} + C_{13}$$

and so H is uniformly bounded from above at (t_0, z_0) . Since $H(0, \cdot) = 0$ and $\varphi_{0,j}$ is both uniformly bounded, H is uniformly bounded for $t \in [0, T]$ and so we prove the lemma. \square

Let $g^{(j)}$ be the Kähler metric associated to ω_j and $\nabla^{(j)}$ be the gradient operator associated to the Kähler form ω_j . As in [Y2], set

$$(\varphi_j)_{p\bar{k}m} = \nabla_p^{(j)} \partial_{\bar{k}} \partial_m \varphi_j$$

and

$$S_j = (g^{(j)})^{p\bar{r}} (g^{(j)})^{s\bar{k}} (g^{(j)})^{m\bar{t}} (\varphi_j)_{p\bar{k}m} (\varphi_j)_{\bar{r}s\bar{t}}.$$

Lemma 3.4 *For any $T < T_0$, there exist constants $\lambda > 0$ and $C > 0$ such that for $t \in [0, T]$,*

$$\|\varphi_j(t, \cdot)\|_{C^3(X)} \leq C e^{\frac{\lambda}{t}}. \quad (3.9)$$

Proof Since all the second order terms are bounded by certain power of $e^{\frac{1}{t}}$. By the computation in [PSS], there exist sufficiently large α and $\beta > 0$ such that

$$(\frac{\partial}{\partial t} - \Delta_j) e^{-\frac{2\alpha}{t}} tr_{\omega_0}(\omega_j) \leq -C_1 e^{-\frac{\alpha}{t}} S_j + C_2$$

and

$$(\frac{\partial}{\partial t} - \Delta_j) e^{-\frac{2\beta}{t}} S_j \leq C_3 e^{-\frac{\beta}{t}} S_j + C_4.$$

By choosing sufficiently large A and $\beta > \alpha$, we have

$$(\frac{\partial}{\partial t} - \Delta_j) (e^{-\frac{2\beta}{t}} S_j + A e^{-\frac{2\alpha}{t}} tr_{\omega_0}(\omega_j)) \leq -C_5 e^{-\frac{\alpha}{t}} S_j - C_6$$

for sufficiently large $A > 0$.

By the maximum principle and Lemma 3.3, $e^{-\frac{\beta}{t}} S_j$ is uniformly bounded for $t \in [0, T]$. \square

Proposition 3.2 *For any $0 < \epsilon < T < T_0$ and $k \geq 0$, there exists $C_{\epsilon, T, k} > 0$ such that,*

$$\|\varphi_j\|_{C^k([\epsilon, T] \times X)} \leq C_{\epsilon, T, k}. \quad (3.10)$$

By Lemma 3.1, φ_j is a Cauchy sequence in $L^\infty([0, T] \times X)$ and so φ_j converges to $\varphi \in L^\infty([0, T] \times X)$ uniformly in $L^\infty([0, T] \times X)$. For any $0 < \delta < T$, φ_j is uniformly bounded in $C^\infty([\delta, T] \times X)$. Therefore φ_j converges to φ in $C^\infty((0, T] \times X)$. Hence $\varphi \in C^\infty((0, T] \times X)$.

Lemma 3.5 $\varphi \in C^0([0, T] \times X)$ and

$$\lim_{t \rightarrow 0^+} \|\varphi(t, \cdot) - \varphi_0(\cdot)\|_{L^\infty(X)} = 0. \quad (3.11)$$

Proof For any $(t, z) \in [0, T] \times X$,

$$|\varphi(t, z) - \varphi_0(z)| \leq |\varphi(t, z) - \varphi_j(t, z)| + |\varphi_j(t, z) - \varphi_{0,j}(z)| + |\varphi_{0,j}(z) - \varphi_0(z)|.$$

Since $\{\varphi_j\}$ is a Cauchy sequence in $L^\infty([0, T] \times X)$,

$$\lim_{j \rightarrow \infty} \|\varphi - \varphi_j\|_{L^\infty([0, T] \times X)} = 0.$$

Also

$$\lim_{j \rightarrow \infty} \|\varphi_{0,j} - \varphi_0\|_{L^\infty(X)} = 0$$

and for any j ,

$$\lim_{t \rightarrow 0^+} \sup_{z \in X} |\varphi_j(t, z) - \varphi_{0,j}(z)| = 0.$$

For any $\epsilon > 0$, there exists $J > 0$ such for any $j > J$,

$$\sup_{(t, z) \in [0, T] \times X} |\varphi(t, z) - \varphi_j(t, z)| < \frac{\epsilon}{3}$$

and

$$\sup_{z \in X} |\varphi_{0,j}(z) - \varphi_0(z)| < \frac{\epsilon}{3}.$$

Fix such j , there exists $0 < \delta < T$ such that

$$\sup_{t \in [0, \delta], z \in X} |\varphi_j(t, z) - \varphi_{0,j}(z)| < \frac{\epsilon}{3}.$$

Combining the above estimates together, for any $t \in [0, \delta]$ and $z \in X$

$$|\varphi(t, z) - \varphi_0(z)| < \epsilon.$$

This completes the proof of the lemma. □

Now we are ready to show the existence and uniqueness for the Monge-Ampère flow starting with $\varphi_0 \in PSH_p(X, \omega_0, \Omega)$.

Proposition 3.3 φ is the unique solution of the following Monge-Ampère equation

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}, & (0, T_0) \times X \\ \varphi(0, \cdot) = \varphi_0 \end{cases} \quad (3.12)$$

in the space of $C^0([0, T_0] \times X) \cap C^\infty((0, T_0) \times X)$.

Proof It suffices to prove the uniqueness. Suppose there exists another solution $\varphi' \in C^0([0, T_0] \times X) \times C^\infty((0, T_0) \times X)$ of the Monge-Ampère flow (3.12).

Let $\psi = \varphi' - \varphi$. Then

$$\begin{cases} \frac{\partial \psi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi + \sqrt{-1}\partial\bar{\partial}\psi)^n}{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}, & (0, T_0) \times X \\ \psi(0, \cdot) = 0. \end{cases} \quad (3.13)$$

By the maximum principle, $\max_X \psi(t, \cdot)$ is decreasing and $\min_X \psi(t, \cdot)$ is increasing on $(0, T_0)$. Since both of $\max_X \psi(t, \cdot)$ and $\min_X \psi(t, \cdot)$ are continuous on $[0, T_0]$ with $\max_X \psi(0, \cdot) = \min_X \psi(0, \cdot) = 0$,

$$\psi(t, \cdot) = 0$$

for $t \in [0, T_0]$. The proposition follows easily. \square

With the above preparations, we can show the smoothing property for the Kähler-Ricci flow with rough initial data.

Theorem 3.1 *Let X be an n -dimensional Kähler manifold. Let ω_0 be a Kähler form and Ω be a smooth volume form on X . Suppose that $\omega'_0 = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0$ for some $\varphi_0 \in PSH_p(\omega_0, \Omega)$ for some $p > 1$. Then there exists a unique family of smooth Kähler metrics $\omega(t, \cdot) \in C^\infty((0, T_0) \times X)$ satisfying the following conditions.*

1. $\frac{\partial \omega}{\partial t} = -Ric(\omega)$, $(0, T_0) \times X$.

2. There exists $\varphi \in C^0([0, T_0] \times X) \cap C^\infty((0, T_0) \times X)$ such that $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ and

$$\lim_{t \rightarrow 0^+} \|\varphi(t, \cdot) - \varphi_0(\cdot)\|_{L^\infty(X)} = 0.$$

In particular, $\omega(t, \cdot)$ converges in the sense of distribution to ω'_0 as $t \rightarrow 0$.

Proof The unnormalized Kähler-Ricci flow $\frac{\partial \omega}{\partial t} = -Ric(\omega)$ is equivalent to the Monge-Ampère flow

$$\sqrt{-1}\partial\bar{\partial}\left(\frac{\partial \varphi}{\partial t} - \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}\right) = 0. \quad (3.14)$$

Then $\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} + f(t)$ with $\lim_{t \rightarrow 0^+} \varphi(t, 0) = \varphi_0$ for a smooth function $f(t)$ on $(0, T_0)$. Proposition 3.3 gives the existence of such a φ with $f(t) = 0$.

Suppose there is another solution $\phi \in C^\infty((0, T_0) \times X) \cap C^0([0, T_0] \times X)$ to the equation (3.14). Then

$$\frac{\partial \phi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\Omega} + f(t)$$

for some smooth function $f(t)$ on $(0, T_0)$. We can assume that $\phi(0, \cdot) = \varphi_0$ by subtracting a constant because $\phi(t, \cdot)$ converges to a continuous $\phi_0(\cdot)$ in $C^0(X)$ as $t \rightarrow 0$, and ϕ_0 differs from φ_0 by a constant. Then we consider the function $\psi = \phi - \varphi$,

$$\frac{\partial \psi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi + \sqrt{-1}\partial\bar{\partial}\psi)^n}{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n} + f(t)$$

with $\psi(0, \cdot) = 0$. By the same argument as that in the proof of Proposition 3.3 we can show that for $0 < t_1 < t_2 < T_0$,

$$\begin{aligned}\max_X \psi(t_2, \cdot) &\leq \max_X \psi(t_1, \cdot) + \int_{t_1}^{t_2} f(t) dt \\ \min_X \psi(t_2, \cdot) &\geq \min_X \psi(t_1, \cdot) + \int_{t_1}^{t_2} f(t) dt\end{aligned}$$

Therefore,

$$\min_X \psi(t_2, \cdot) \geq \max_X \psi(t_2, \cdot) - (\max_X \psi(t_1, \cdot) - \min_X \psi(t_1, \cdot)).$$

and by letting $t_1 \rightarrow 0^+$, we have

$$\min_X \psi(t_2, \cdot) \geq \max_X \psi(t_2, \cdot).$$

So $\psi(t, \cdot) = \psi(t) = \int_0^t f(s) ds$.

□

Theorem 3.1 shows that the Kähler-Ricci flow smooths out the initial semi-positive closed $(1,1)$ -current ω with bounded local potential and $\omega^n \in L^p(X)$ for some $p > 1$. It improves a result of Chen-Tian-Zhang [CTZ] (also see [CT] and [CD]), where $p > 3$. We remark that the condition that $p > 1$ is essential for later estimates and geometric applications.

3.2 Monge-Ampère flows with degenerate initial data

In this section, we will investigate a family of Monge-Ampère flows with singular data on a smooth projective variety. The existence and uniqueness for the solutions will be proved.

We start with two conditions prescribing the singularity and degeneracy of the data that will be considered along certain Monge-Ampère flows. These conditions arise naturally in the geometric setting in later discussions.

Condition A. Let X be an n -dimensional projective manifold. Let $L_1 \rightarrow X$ be a big and semi-ample line bundle over X and $L_2 \rightarrow X$ be a line bundle such that $[L_1 + \epsilon L_2]$ is still semi-ample for $\epsilon > 0$ sufficiently small. Let $\omega_0 \in c_1(L_1)$ be a smooth semi-positive closed $(1,1)$ -form on X and $\chi \in c_1(L_2)$ a smooth closed $(1,1)$ -form. Let $\omega_t = \omega_0 + t\chi$. We assume that ω_0 at worst vanishes along a projective subvariety of X to a finite order, that is, there exists an effective divisor E_0 on X such that for any fixed Kähler metric ϑ ,

$$\omega_0 \geq C_\vartheta |S_{E_0}|_{h_{E_0}}^2 \vartheta,$$

where $C_\vartheta > 0$ is a constant, S_{E_0} is a defining section of E_0 and h_{E_0} is a smooth hermitian metric on the line bundle associated to E_0 .

Such an ω_0 always exists. For example, let m be sufficiently large such that $(L_1)^m$ is globally generated and let $\{S_j^{(m)}\}_{j=0}^{d_m}$ be a basis of $H^0(X, (L_1)^m)$. We can then let

$$\omega_0 = \frac{1}{m} \sqrt{-1} \partial \bar{\partial} \log \sum_{j=0}^{d_m} |S_j^{(m)}|^2.$$

Condition B. Let Θ be a smooth volume form on X . Let $E = \sum_{i=1}^p a_i E_i$ and $F = \sum_{j=1}^q b_j F_j$ be effective divisors on X , where E_i and F_j are irreducible components of E with simple normal crossings. In addition, we assume $a_i \geq 0$ and $0 < b_j < 1$. Let Ω be a semi-positive (n, n) -form on X such that $\int_X \Omega > 0$ and

$$\Omega = |S_E|_{h_E}^2 |S_F|_{h_F}^{-2} \Theta, \quad (3.15)$$

where S_E and S_F are the multi-valued holomorphic defining sections of E and F , h_E and h_F are smooth hermitian metrics on the line bundles associated to E and F .

Note that the condition $b_j \in (0, 1)$ makes Ω an integrable (n, n) -form on X . Furthermore, $\frac{\Omega}{\Theta}$ is in $L^p(X, \Theta)$ for some $p > 1$.

Since L_1 is big and semi-ample, by Kodaira's lemma, there exists an effective \mathbf{Q} -divisor \tilde{E} such that $[L_1] - \epsilon[\tilde{E}]$ is ample for any sufficiently small rational $\epsilon > 0$. Without loss of generality, we can always assume that the support of \tilde{E} contains E_0 , E and F , i.e.,

$$\text{supp}E \cup \text{supp}F \cup \text{supp}E_0 \subset \text{supp}\tilde{E}.$$

Let $S_{\tilde{E}}$ be the defining section of \tilde{E} and $h_{\tilde{E}}$ a smooth hermitian metric on the line bundle associated to $[\tilde{E}]$ such that for sufficiently small $\epsilon > 0$,

$$\omega_0 - \epsilon \text{Ric}(h_{\tilde{E}}) > 0.$$

We can also scale $h_{\tilde{E}}$ and assume $|S_{\tilde{E}}|_{h_{\tilde{E}}}^2 \leq 1$ on X .

Let $\omega_t = \omega_0 + t\chi$. We consider the following Monge-Ampère flow with the initial data $\varphi_0 \in PSH(X, \omega_0) \cap C^\infty(X)$.

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_0 + t\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}, \\ \varphi(0, \cdot) = \varphi_0. \end{cases} \quad (3.16)$$

The equation (3.16) is not only degenerate in the sense that $[\omega_t]$ is not necessarily Kähler but also that Ω has zeros and poles along E and F . The goal of the following discussion is to prove the existence and uniqueness of the solution for the Monge-Ampère flow (3.16) with appropriate assumptions. Let

$$T_0 = \sup\{t \geq 0 \mid [L_1 + tL_2] \text{ is semi-ample }\}.$$

If **Condition A** is satisfied, $T_0 > 0$ or $T_0 = \infty$. Furthermore, $L_1 + tL_2$ is big for any $t \in [0, T_0)$.

The following theorem is the main result of this section.

Theorem 3.2 *Let X be an n -dimensional projective manifold. Suppose **Condition A** and **Condition B** are satisfied. Then for any $\varphi_0 \in PSH(X, \omega_0) \cap C^\infty(X)$, there exists a unique $\varphi \in C^\infty([0, T_0)) \times (X \setminus \tilde{E})$ with $\varphi(t, \cdot) \in PSH(X, \omega_t) \cap L^\infty(X)$ for each $t \in [0, T_0)$, satisfying the following Monge-Ampère flow.*

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}, & \text{on } [0, T_0) \times X \setminus \tilde{E}, \\ \varphi(0, \cdot) = \varphi_0, & \text{on } X. \end{cases} \quad (3.17)$$

Remark 3.1 For any fixed $T \in (0, T_0)$, we can assume that $\omega_t \geq \epsilon \omega_0$ for all $t \in [0, T]$, where $\epsilon > 0$ is sufficiently small and it depends on T . Recall that $L_1 + tL_2$ is semi-ample and big for all $t \in [0, T_0)$. We fix $T' \in (T, T_0)$. Then

$$[\omega_t] = \frac{t}{T_0} [\omega_0 + T_0 \chi] + \frac{T_0 - t}{T_0} [\omega_0]$$

and $[\omega_0 + T_0 \chi]$ is semi-positive and big. Then there exists $\phi \in PSH(X, \omega_0 + T_0 \chi) \cap C^\infty(X)$, i.e., $\omega_0 + T_0 \chi + \sqrt{-1} \partial \bar{\partial} \phi \geq 0$. Then for all $t \in [0, T']$,

$$\omega_t + \frac{t}{T_0} \sqrt{-1} \partial \bar{\partial} \phi = \frac{t}{T_0} (\omega_0 + T_0 \chi) + \frac{t}{T_0} \sqrt{-1} \partial \bar{\partial} \phi + \frac{T_0 - t}{T_0} \omega_0 \geq \frac{T_0 - t}{T_0} \omega_0 \geq 0.$$

Then

$$\frac{\partial}{\partial t} \left(\varphi - \frac{t}{T_0} \phi \right) = \log \frac{(\omega_0 + t(\chi + \frac{1}{T_0} \sqrt{-1} \partial \bar{\partial} \phi)) + \sqrt{-1} \partial \bar{\partial} (\varphi - \frac{t}{T_0} \phi))^n}{\Omega} - \frac{1}{T_0} \phi.$$

Let $\omega'_0 = \omega_0$, $\chi' = \chi + \frac{1}{T_0} \sqrt{-1} \partial \bar{\partial} \phi$, $\omega'_t = \omega'_0 + t\chi'$, $\Omega' = \Omega e^{\frac{\phi}{T_0}}$ and $\varphi' = \varphi - \frac{t}{T_0} \phi$. We have

$$\frac{\partial}{\partial t} \varphi' = \log \frac{(\omega'_t + \sqrt{-1} \partial \bar{\partial} \varphi')^n}{\Omega'}.$$

It is easy to check that $\omega'_t \geq \frac{T_0 - t}{T_0} \omega_0$ and **Condition A** and **Condition B** are still satisfied for ω'_0 and Ω' .

From now on, we fix $T \in (0, T_0)$ and assume without loss of generality from the previous remark that ω_t is bounded from below by $\epsilon \omega_0$ for sufficiently small $\epsilon > 0$. In order to prove Theorem 3.2, we have to perturb equation (3.17) in order to obtain smooth approximating solutions. Let

$$\omega_{t,s} = \omega_0 + t\chi + s\vartheta$$

and

$$\Omega_{w,r} = \frac{r + |S_E|_{h_E}^2}{w + |S_F|_{h_F}^2} \Theta$$

be the perturbations of ω_t and Ω for $s, w, r \in [0, 1]$. In particular, $\omega_{t,0} = \omega_t$ and $\Omega_{0,0} = \Omega$. Since ϑ is Kähler, $\omega_{t,s}$ is Kähler for $s > 0$ and $t \in [0, T_0)$.

Then we consider the following well-defined family of Monge-Ampère flows with the fixed initial data φ_0 .

$$\begin{cases} \frac{\partial \varphi_{s,w,r}}{\partial t} = \log \frac{(\omega_{s,t} + \sqrt{-1} \partial \bar{\partial} \varphi_{s,w,r})^n}{\Omega_{w,r}}, \\ \varphi_{s,w,r}(0, \cdot) = \varphi_0. \end{cases} \quad (3.18)$$

The standard argument gives the following lemma as $[\omega_{s,t}]$ stays positive on $[0, T_0)$.

Lemma 3.6 *For any $s, w, r \in (0, 1]$, there is a unique smooth solution $\varphi_{s,w,r}$ of the Monge-Ampère flow (3.18) on $[0, T_0) \times X$.*

Lemma 3.7 Let $F_{w,r} = \frac{\Omega_{w,r}}{\vartheta^n}$. Then there exist constants $p > 1$ and $C > 0$ such that for $w, r \in [0, 1]$ and

$$\|F_{w,r}\|_{L^p(X, \vartheta^n)} \leq C. \quad (3.19)$$

Proof There exists a constant $C > 0$ such that

$$\begin{aligned} \int_X (F_{w,r})^p \vartheta^n &= \int_X (F_{w,r})^{p-1} \Omega_{w,r} \\ &= \int_X (F_{w,r})^{p-1} (r + |S_E|)_{h_E}^2 (w + |S_F|)_{h_F}^{-2} \Theta \\ &\leq C \int_X (F_{w,r})^{p-1} |S_F|_{h_F}^{-2} \Theta. \end{aligned}$$

Since $F_{w,r}$ has at worst poles along \tilde{E} and the vanishing order of $|S_F|_{h_F}^2$ is strictly less than 2, by choosing $p-1 > 0$ sufficiently small, $\int_X (F_{w,r})^p \vartheta^n$ is uniformly bounded from above. \square

We can apply the results for degenerate complex Monge-Ampère equations as $F_{w,r}$ is uniformly bounded in $L^p(X)$ for some $p > 1$.

Lemma 3.8 For any $0 < T < T_0$, there exists $C > 0$ such that for all $s, w, r \in (0, 1]$,

$$\|\varphi_{s,r,w}\|_{L^\infty([0,T] \times X)} \leq C$$

Proof We first prove the uniform upper bound for $\varphi_{s,w,r}$. We define for $t \in [0, T]$

$$\alpha_{s,w,r}(t) = \frac{\int_X \Omega_{w,r}}{[\omega_{t,s}]^n}. \quad (3.20)$$

It is easy to see that $\alpha_{s,w,r}(t)$ is uniformly bounded for $t \in [0, T]$. Notice that

$$\begin{aligned} \frac{\partial}{\partial t} \int_X \varphi_{s,w,r} \Omega_{w,r} &= \int_X \log \frac{(\omega_{t,s} + \sqrt{-1}\partial\bar{\partial}\varphi_{s,w,r})^n}{\Omega_{w,r}} \Omega_{w,r} \\ &\leq \left(\int_X \Omega_{w,r} \right) \left(\log \frac{\int_X (\omega_{t,s} + \sqrt{-1}\partial\bar{\partial}\varphi_{s,w,r})^n}{\alpha_{s,w,r}(t) \int_X \Omega_{w,r}} \right) + \alpha_{s,w,r}(t) \int_X \Omega_{w,r} \\ &\leq \left(\int_X \Omega_{w,r} \right) \left(\log \frac{[\omega_{t,s}]^n}{\alpha_{s,w,r}(t) \int_X \Omega_{w,r}} \right) + \alpha_{s,w,r}(t) \int_X \Omega_{w,r} \\ &= \alpha_{s,w,r}(t) \int_X \Omega_{w,r}. \end{aligned}$$

By the maximum principle, $\int_X \varphi_{s,w,r} \Omega_{w,r} \leq C$ for a uniform constant C that depends on T .

On the other hand, since $\varphi_{s,w,r} \in PSH(X, \omega_{s,w,r})$, by Hörmander-Tian's estimate, there exist $\alpha > 0$ and $C_\alpha > 0$ such that for all $s, r \in (0, 1]$ and $t \in [0, T]$,

$$\int_X e^{-\alpha(\varphi_{s,w,r} - \sup_X \varphi_{s,w,r})} \Omega_{w,r} \leq C_\alpha.$$

By Jensen's inequality, there exists a constant $C > 0$ such that for $s, r \in (0, 1]$ and $t \in [0, T]$,

$$\sup_X \varphi_{s,w,r} - \int_{\tilde{X}} \varphi_{s,w,r} \Omega_{w,r} \leq C.$$

It follows that $\sup_X \varphi_{s,w,r}$ is uniformly bounded above.

Now it suffices to obtain a uniform lower bound for $\varphi_{s,w,r}$. Let $\theta = \delta\omega_0$, where $\delta > 0$ is sufficiently small such that

$$2\theta \leq \omega_t$$

for all $t \in [0, T]$. By the choice of $h_{\tilde{E}}$,

$$\theta - \epsilon Ric(h_{\tilde{E}}) > 0$$

for any sufficiently small $\epsilon > 0$.

We consider the following family of Monge-Ampère equations for $w, r \in [0, 1]$.

$$(\theta + \sqrt{-1}\partial\bar{\partial}\phi_{w,r})^n = C_{w,r}\Omega_{w,r}, \quad (3.21)$$

with the normalization conditions $[\theta]^n = C_{w,r} \int_X \Omega_{w,r}$ and $\sup_X \phi = 0$.

Note that by Lemma 3.7, $\frac{C_{w,r}\Omega_{w,r}}{\theta^n}$ is uniformly bounded in $L^p(X, \theta^n)$ for $w, r \in [0, 1]$. By the results in [EGZ1], $\phi_{w,r} \in C^\infty(X \setminus \tilde{E})$ and there exists a uniform constant $C > 0$ such that for all $w, r \in [0, 1]$,

$$\|\phi_{w,r}\|_{L^\infty(X)} \leq C. \quad (3.22)$$

Let $\psi_{s,w,r}(t, \cdot) = \varphi_{s,w,r}(t, \cdot) - \phi_{w,r}$. The evolution equation for $\psi_{s,w,r}$ is given by the following formula for $t \in [0, T]$.

$$\frac{\partial}{\partial t} \psi_{s,w,r} = \log \frac{(\omega_{t,s} + \sqrt{-1}\partial\bar{\partial}\varphi_{s,r})^n}{(\theta + \sqrt{-1}\partial\bar{\partial}\phi_{w,r})^n} + \log C_{w,r}.$$

Let $H = \psi_{s,w,r} - \epsilon \log |S_{\tilde{E}}|^2_{h_{\tilde{E}}}$. Then the minimum of H is always achieved in $X \setminus \tilde{E}$ as H tends to ∞ near \tilde{E} . Suppose for fixed $t \in [0, T]$, $\min_X H(t, \cdot) = H(t, z_0)$. By choosing sufficiently small $\epsilon > 0$, we have at (t, z_0) ,

$$\begin{aligned} \frac{\partial}{\partial t} H &= \log \frac{(\theta + \sqrt{-1}\partial\bar{\partial}\phi_{w,r} + (\omega_t - \theta + s\vartheta - \epsilon Ric(h_{\tilde{E}}) + \sqrt{-1}\partial\bar{\partial}H)^n)^n}{(\theta + \sqrt{-1}\partial\bar{\partial}\phi_{w,r})^n} + \log C_{w,r} \\ &\geq \log \frac{(\theta + \sqrt{-1}\partial\bar{\partial}\phi_{w,r} + (\theta - \epsilon Ric(h_{\tilde{E}})) + \sqrt{-1}\partial\bar{\partial}H)^n}{(\theta + \sqrt{-1}\partial\bar{\partial}\phi_{w,r})^n} + \log C_{w,r} \\ &\geq \log \frac{(\theta + \sqrt{-1}\partial\bar{\partial}\phi_{w,r} + \sqrt{-1}\partial\bar{\partial}H)^n}{(\theta + \sqrt{-1}\partial\bar{\partial}\phi_{w,r})^n} + \log C_{w,r} \\ &\geq \log C_{w,r}. \end{aligned}$$

By the maximum principle, there exist a sufficiently small $\epsilon > 0$ and a uniform constant $C > 0$ independent of ϵ such that for all $s, w, r \in (0, 1]$, $t \in [0, T]$,

$$H \geq -C.$$

It follows then that for $s, w, r \in (0, 1]$ and $t \in [0, T]$,

$$\varphi_{s,w,r}(t, \cdot) \geq \phi_{w,r}(\cdot) + \epsilon \log |S_{\tilde{E}}|^2_{h_{\tilde{E}}} - C.$$

By letting $\epsilon \rightarrow 0$, we have

$$\varphi_{s,w,r}(t, \cdot) \geq \phi_{w,r}(\cdot) - C.$$

The uniform bound for $\phi_{w,r}$ gives the uniform lower bound for $\varphi_{s,w,r}$. Combined with the upper bound for $\varphi_{s,w,r}$, we have completed the proof. \square

Lemma 3.9 *For any $T \in (0, T_0)$, there exist $C, \alpha > 0$ such that for all $t \in [0, T]$ and $s, w, r \in (0, 1]$*

$$\left| \frac{\partial}{\partial t} \varphi_{s,w,r} \right| \leq C + \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^{-2\alpha}.$$

Proof Let $\Delta_{s,w,r}$ be the Laplace operator with respect to the Kähler metric $\omega_{s,w,r}$. Notice that

$$\frac{\partial}{\partial t} \dot{\varphi}_{s,w,r} = \Delta_{s,w,r} \dot{\varphi}_{s,w,r} + \text{tr}_{\omega_{s,w,r}}(\chi).$$

Let $H^+ = \left(\dot{\varphi}_{s,w,r} - A^2 \varphi_{s,w,r} + A \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 \right)$. $H^+(0, \cdot)$ is uniformly bounded from above for $A > 0$ sufficiently large. Then for A sufficiently large, we have

$$\begin{aligned} & \frac{\partial}{\partial t} H^+ \\ &= \Delta_{s,w,r} H^+ - \text{tr}_{\omega_{s,w,r}}(A^2 \omega_{t,s} - \chi - A \text{Ric}(h_{\tilde{E}})) - A^2 \dot{\varphi}_{s,w,r} + nA^2 \\ &\leq \Delta_{s,w,r} H^+ - A^2 \dot{\varphi}_{s,w,r} + nA^2 \\ &= \Delta_{s,w,r} H^+ - A^2 H^+ + A^2(-A^2 \varphi_{s,w,r} + A \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2) + nA^2 \\ &\leq \Delta_{s,w,r} H^+ - A^2 H^+ + C. \end{aligned}$$

By the maximum principle, H^+ is uniformly bounded above and so there exist C_1 and C_2 such that

$$\dot{\varphi}_{s,w,r} \leq C_1 + C_2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2.$$

To estimate the lower bound of $\dot{\varphi}_{s,w,r}$, we define

$$H^- = \dot{\varphi}_{s,w,r} + A^2 \varphi_{s,w,r} - A \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2$$

for sufficiently large A . Then straightforward calculation shows that there exist constants C_3, C_4, \dots, C_7 such that

$$\begin{aligned} \frac{\partial}{\partial t} H^- &\geq \Delta_{s,w,r} H^- + C_3 \left(\frac{\omega_{t,s}^n}{\omega_{s,w,r}^n} \right)^{\frac{1}{n}} + A^2 \dot{\varphi}_{s,w,r} - C_4 \\ &= \Delta_{s,w,r} H^- + C_3 \left(\frac{\omega_{t,s}^n}{\omega_{s,w,r}^n} \right)^{\frac{1}{n}} + A^2 \log \frac{\omega_{s,w,r}^n}{\omega_{t,s}^n} + A^2 \log \frac{\omega_{t,s}^n}{\Omega_{w,r}} - C_4 \\ &\geq \Delta_{s,w,r} H^- - A^2 \log \frac{\omega_{s,w,r}^n}{\omega_{t,s}^n} + A^2 \log \frac{\omega_{t,s}^n}{\Omega_{w,r}} - C_5 \\ &= \Delta_{s,w,r} H^- - A^2 H^- - A^3 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 + 2A^2 \log \frac{\omega_{t,s}^n}{\Omega_{w,r}} - C_6 \\ &\geq \Delta_{s,w,r} H^- - A^2 H^- - C_7. \end{aligned}$$

Then a similar argument by the maximum principle gives the lower bound for H^- and $\dot{\varphi}_{s,w,r}$. \square

Lemma 3.10 For any $T \in (0, T_0)$, there exist $C, \alpha > 0$ such that for all $t \in [0, T]$ and $s, r, w \in (0, 1]$

$$|tr_\vartheta(\omega_{s,w,r})| \leq C|S_{\tilde{E}}|_{h_{\tilde{E}}}^{-2\alpha}.$$

Proof Standard calculations show that for some constant $C > 0$,

$$\left(\frac{\partial}{\partial t} - \Delta_{s,w,r}\right) \log tr_\vartheta(\omega_{s,w,r}) \leq C tr_{\omega_{s,w,r}}(\vartheta) + \frac{tr_{\omega_{s,w,r}}(Ric(\Omega_{w,r}))}{tr_{\omega_{s,w,r}}(\vartheta)} + C.$$

Define

$$H = \log tr_\vartheta(\omega_{s,w,r}) - A^2 \varphi_{s,w,r} + A \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2.$$

Then for sufficiently large $A > 0$, there exist uniform constants C_1 and C_2 such that

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{s,w,r}\right) H \\ & \leq -tr_{\omega_{s,w,r}}(A^2 \omega_{t,s} - ARic(h_{\tilde{E}}) - C\vartheta) + \frac{tr_{\omega_{s,w,r}}(Ric(\Omega_{w,r}))}{tr_{\omega_{s,w,r}}(\vartheta)} - A^2 \log \frac{\omega_{s,w,r}^n}{\Omega_{w,r}} + C_1 \\ & \leq -A tr_{\omega_{s,w,r}}(\theta) + \frac{tr_{\omega_{s,w,r}}(Ric(\Omega_{w,r}))}{tr_{\omega_{s,w,r}}(\vartheta)} - A^2 \log \frac{\omega_{s,w,r}^n}{\Omega_{w,r}} + C_2. \end{aligned}$$

Suppose $\max_{[0,t] \times X} H = H(t_0, z_0)$. Then $z_0 \in X \setminus \tilde{E}$ and at (t_0, z_0) , there exist α_1, C_3 and C_4 such that

$$tr_{\omega_{s,w,r}}(\theta) \leq A^{-1} \frac{tr_{\omega_{s,w,r}}(A Ric(\Omega_{w,r}))}{tr_{\omega_{s,w,r}}(\vartheta)} - A \log \frac{\omega_{s,w,r}^n}{\Omega_{w,r}} + C_3 \leq C_4 |S_{\tilde{E}}|_{h_{\tilde{E}}}^{-2\alpha_1}.$$

Applying the mean value inequality and Lemma 3.9, there exist α_2 and C_5

$$tr_\vartheta(\omega_{s,w,r}) \leq C_5 |S_{\tilde{E}}|_{h_{\tilde{E}}}^{-2\alpha_2}.$$

Therefore $H(t_0, z_0)$ is uniformly bounded from above. The lemma then follows easily. \square

The following proposition gives a uniform bound for the approximating Kähler metrics $\omega_{s,w,r}$ away from \tilde{E} .

Proposition 3.4 For any $T \in (0, T_0)$, $K \subset\subset X \setminus \tilde{E}$ and $k > 0$, there exists $C_{k,K,T}$ such that

$$\|\varphi_{s,w,r}\|_{C^k([0,T] \times K)} \leq C_{k,K,T}.$$

Proof The proof follows from standard Schauder's estimates. \square

Our goal is to construct a solution by the approximating solutions $\varphi_{s,w,r}$.

Lemma 3.11 The following monotonicity conditions hold for $\varphi_{s,w,r}$.

1. For any $0 < r_1 \leq r_2 \leq 1$ and $s, w \in (0, 1]$,

$$\varphi_{s,w,r_1} \geq \varphi_{s,w,r_2}.$$

2. For any $0 < w_1 \leq w_2 \leq 1$ and $s, r \in (0, 1]$,

$$\varphi_{s,w_1,r} \leq \varphi_{s,w_2,r}.$$

3. For any $0 < s_1 \leq s_2 \leq 1$ and $w, r \in (0, 1]$,

$$\varphi_{s_1,w,r} \leq \varphi_{s_2,w,r}.$$

Proof The proof is a straightforward application of the maximum principle. \square

Fix $T \in (0, T_0)$, for each $t \in [0, T]$, let

$$\varphi_{s,w}(t, \cdot) = (\limsup_{r \rightarrow 0} \varphi_{s,w,r}(t, \cdot))^*$$

where $f^*(z) = \lim_{\delta \rightarrow 0} \sup_{B_\delta(z)} f(\cdot)$. Then $\varphi_{s,w} \in PSH(X, \omega_{t,s}) \cap L^\infty(X) \cap C^\infty(X \setminus \tilde{E})$ and $\varphi_{s,w,r}$ converges to $\varphi_{s,w}$ on $X \setminus \tilde{E}$ locally in C^∞ -topology by estimates from Lemma 3.8 and Proposition 3.4. The following monotonicity also holds and follows easily from the above results.

Lemma 3.12 For any $0 < s_1 \leq s_2 \leq 1$ and $w \in (0, 1]$,

$$\varphi_{s_1,w} \leq \varphi_{s_2,w}.$$

Also for any $0 < w_1 \leq w_2 \leq 1$ and $s \in (0, 1]$,

$$\varphi_{s,w_1} \leq \varphi_{s,w_2}.$$

Furthermore, for any $T \in (0, T_0)$, $K \subset\subset X \setminus \tilde{E}$ and $k > 0$, there exists $C_{K,k,T} > 0$ such that on $[0, T]$,

$$\|\varphi_{s,w}\|_{C^k(K)} \leq C_{K,k,T}.$$

Let

$$\varphi_s(t, \cdot) = \lim_{w \rightarrow 0} \varphi_{s,w}(t, \cdot).$$

Then $\varphi_s \in PSH(X, \omega_{t,s}) \cap L^\infty(X) \cap C^\infty(X \setminus \tilde{E})$ and $\varphi_{s,w}$ converges to φ_s on $X \setminus \tilde{E}$ locally in C^∞ -topology.

Lemma 3.13 For any $0 < s_1 \leq s_2 \leq 1$,

$$\varphi_{s_1} \leq \varphi_{s_2}.$$

Furthermore, for any $T \in (0, T_0)$, $K \subset\subset X \setminus \tilde{E}$ and $k > 0$, there exists $C_{K,k,T} > 0$ such that such that on $[0, T]$,

$$\|\varphi_s\|_{C^k(K)} \leq C_{K,k,T}.$$

Let $\varphi = \lim_{s \rightarrow 0} \varphi_s$. Since φ_s is decreasing as $s \rightarrow 0$ is and φ_s is bounded below uniformly, we have

$$\varphi \in PSH(X, \omega_{t,0}) \cap L^\infty(X) \cap C^\infty(X \setminus \tilde{E}).$$

Furthermore, for any $K \subset\subset X \setminus \tilde{E}$,

$$\varphi_s \rightarrow \varphi$$

in $C^\infty([0, T] \times K)$. The following corollary is then immediate.

Corollary 3.1 φ satisfies the following Monge-Ampère flow

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}, & \text{on } [0, T] \times (X \setminus \tilde{E}) \\ \varphi(0, \cdot) = \varphi_0, & \text{on } X. \end{cases} \quad (3.23)$$

In order to prove the uniqueness of the solution of the Monge-Ampère flow (3.23), we consider a family of new Monge-Ampère flows with one more parameter.

Let $\omega_{t,s}^{(\delta)} = (1 - \delta)\omega_0 + t\chi + s\vartheta = \omega_{t,s} - \delta\omega_0$. Then for fixed $T \in [0, T_0]$, there exists $\delta_0 > 0$, such that $(1 - \delta)\omega_0 + t\chi \geq 0$ for all $t \in [0, T]$ and $\delta \in [-\delta_0, \delta_0]$.

The following family of Monge-Ampère flows admit smooth solutions in $C^\infty([0, T] \times X)$

$$\begin{cases} \frac{\partial}{\partial t} \varphi_{s,w,r}^{(\delta)} = \log \frac{(\omega_{t,s}^{(\delta)} + \sqrt{-1}\partial\bar{\partial}\varphi_{s,w,r}^{(\delta)})^n}{\Omega_{w,r}} \\ \varphi_{s,w,r}^{(\delta)}(0, \cdot) = (1 - \delta)\varphi_0, \end{cases} \quad (3.24)$$

Obviously for $s, w, r \in (0, 1]$, $\varphi_{s,w,r}^{(\delta)}$ is a smooth family in δ and

$$\lim_{\delta \rightarrow 0} \varphi_{s,w,r}^{(\delta)} = \varphi_{s,w,r}$$

in C^∞ -topology.

Lemma 3.14 For any $T \in (0, T_0)$, there exist C and $\delta_0 > 0$ such that for $s, w, r \in (0, 1]$, $t \in [0, T]$ and $\delta \in [-\delta_0, \delta_0]$,

$$C \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - C \leq \frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)} \leq C.$$

Proof Let $\Delta_{s,w,r}^{(\delta)}$ be the Laplace operator with respect to $\omega_{s,w,r}^{(\delta)} = \omega_{t,s}^{(\delta)} + \sqrt{-1}\partial\bar{\partial}\varphi_{s,w,r}^{(\delta)}$. Notice that $\frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)} = -\varphi_0$ when $t = 0$ and

$$\left(\frac{\partial}{\partial t} - \Delta_{s,w,r}^{(\delta)} \right) \left(\frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)} \right) = -\text{tr}_{\omega_{s,w,r}^{(\delta)}}(\omega_0) \leq 0.$$

It is easy to see that $\frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)}$ is uniformly bounded above by the maximum principle. By the similar argument as before, $\varphi_{s,w,r}^{(\delta)}$ is bounded in $L^\infty(X)$ uniformly in s, w, r, δ and $t \in [0, T]$.

Consider $H = e^{-A^2 t} \frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)} + A^2 \varphi_{s,w,r}^{(\delta)} - A \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2$. H is uniformly bounded when $t = 0$. There exist constants C_1, C_2, C_3 and $C_4 > 0$ such that

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{s,w,r}^{(\delta)} \right) H \\ = & A^2 \frac{\partial}{\partial t} \varphi_{s,w,r}^{(\delta)} + \text{tr}_{\omega_{s,w,r}^{(\delta)}}(A^2 \omega_{t,s}^{(\delta)} - A \text{Ric}(h_{\tilde{E}}) - e^{-A^2 t} \omega_0) - nA^2 - A^2 e^{-A^2 t} \frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)} \\ \geq & -A^2 \log \frac{\Omega_{w,r}}{(\omega_{s,w,r}^{(\delta)})^n} + C_1 \text{tr}_{\omega_{s,w,r}^{(\delta)}}(\omega_{t,s}^{(\delta)}) - A^2 H - A^3 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - C_2 \\ \geq & -A^2 \log \frac{\Omega_{w,r}}{(\omega_{t,s}^{(\delta)})^n} - A^3 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - A^2 H - C_3 \\ \geq & -A^2 H - C_4. \end{aligned}$$

Therefore H is uniformly bounded from below by the maximum principle and the lemma easily follows. \square

By the same argument as for $\delta = 0$, for $s, \delta \in (0, 1]$, $\lim_{w \rightarrow 0} (\limsup_{r \rightarrow 0} \varphi_{s,w,r}^{(\delta)})^*$ exists on $X \setminus \tilde{E}$ and there exists $\varphi_s^{(\delta)} \in L^\infty([0, T] \times X) \cap C^\infty([0, T] \times X \setminus \tilde{E})$ such that

$$\varphi_s^{(\delta)} = \lim_{w \rightarrow 0} (\limsup_{r \rightarrow 0} \varphi_{s,w,r}^{(\delta)})^*$$

and it satisfies the following Monge-Ampère equation

$$\begin{cases} \frac{\partial}{\partial t} \varphi_s^{(\delta)} = \log \frac{(\omega_{t,s}^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi_s^{(\delta)})^n}{\Omega}, & \text{on } X \setminus \tilde{E} \\ \varphi_s^{(\delta)}(0, \cdot) = (1 - \delta) \varphi_0. \end{cases} \quad (3.25)$$

Let

$$\varphi^{(\delta)} = \lim_{s \rightarrow 0} \varphi_s^{(\delta)}$$

as $\varphi_s^{(\delta)}$ is decreasing as $s \rightarrow 0$. Then $\varphi^{(\delta)} \in L^\infty([0, T] \times X) \cap C^\infty([0, T] \times X \setminus \tilde{E})$ solves the following Monge-Ampère equation.

$$\begin{cases} \frac{\partial}{\partial t} \varphi^{(\delta)} = \log \frac{(\omega_t^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi^{(\delta)})^n}{\Omega}, & \text{on } [0, T] \times X \setminus \tilde{E} \\ \varphi^{(\delta)}(0, \cdot) = (1 - \delta) \varphi_0, \end{cases} \quad (3.26)$$

where $\omega_t^{(\delta)} = \omega_{t,0}^{(\delta)} = (1 - \delta) \omega_0 + t \chi$.

Lemma 3.15 *For any $T \in (0, T_0)$, there exist C and $\delta_0 > 0$ such that on $[0, T] \times X$, for all $s \in (0, 1]$ and $\delta_1, \delta_2 \in [-\delta_0, \delta_0]$,*

$$|\varphi_s^{(\delta_1)} - \varphi_s^{(\delta_2)}| \leq C |\delta_1 - \delta_2| (1 - \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2), \quad (3.27)$$

and so

$$|\varphi^{(\delta_1)} - \varphi^{(\delta_2)}| \leq C |\delta_1 - \delta_2| (1 - \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2) \quad (3.28)$$

Proof This is an immediate result of Lemma 3.14 by letting $w, r \rightarrow 0$ and then $s \rightarrow 0$. \square

Corollary 3.2 *For any $T \in (0, T_0)$ and $K \subset\subset X \setminus \tilde{E}$, $\varphi^{(\delta)}$ converges to φ uniformly in $L^\infty([0, T] \times K)$ as $\delta \rightarrow 0$.*

Proof By Lemma 3.15, $\varphi_s^{(\delta)}$ is uniformly Lipschitz in δ on K and $s \in [0, 1)$. The corollary follows easily by letting $\delta \rightarrow 0$. \square

Now we are able to prove our main result for the existence and uniqueness of the Monge-Ampère solution.

Proof of Theorem 3.2

For any $T \in [0, T_0]$, Corollary 3.1 gives the existence of the solution for the Monge-Ampère flow (3.17). By Lemma 3.9, $\frac{\partial \varphi}{\partial t}$ is integrable and so $\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}$ as distributions. Now it suffices to prove the uniqueness for the solution on $[0, T] \times X$ for any $0 < T < T_0$.

Suppose there is another solution φ' satisfying the Monge-Ampère flow (3.17) such that $\varphi' \in C^\infty([0, T]) \times X \setminus \tilde{E}$ and $\varphi(t, \cdot) \in PSH(X, \omega_t) \cap L^\infty(X)$ for each $t \in [0, T]$.

First, we show that

$$\varphi' \leq \varphi.$$

Let $\psi_{s,\epsilon} = \varphi_s - \varphi' - \epsilon s \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2$ for sufficiently small $\epsilon > 0$. Then $\psi_\epsilon \in C^\infty([0, T] \times (X \setminus \tilde{E}))$ and

$$\frac{\partial}{\partial t} \psi_{s,\epsilon} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi' + s(\vartheta - \epsilon Ric(h_{\tilde{E}})) + \sqrt{-1}\partial\bar{\partial}\psi_{s,\epsilon})^n}{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi')^n}.$$

Suppose $\psi_{s,\epsilon}(t, z_{min}) = \min_X \psi_{s,\epsilon}(t, \cdot)$. Then $z_{min} \in X \setminus \tilde{E}$ since both φ_s and $\varphi' \in L^\infty(X)$. If we choose ϵ sufficiently small, then by the maximum principle,

$$\frac{\partial}{\partial t} \psi_{s,\epsilon}(t, z_{min}) \geq \log \frac{(\omega_t(t, z_{min}) + \sqrt{-1}\partial\bar{\partial}\varphi'(t, z_{min}) + \sqrt{-1}\partial\bar{\partial}\psi_{s,\epsilon}(t, z_{min}))^n}{(\omega_t(t, z_{min}) + \sqrt{-1}\partial\bar{\partial}\varphi'(t, z_{min}))^n} \geq 0.$$

Note that $\psi_{s,\epsilon}(0, \cdot) = -\epsilon s \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 \geq 0$ and so

$$\psi_{s,\epsilon} \geq 0$$

for any ϵ sufficiently small. Therefore by letting $\epsilon \rightarrow 0$, we have

$$\varphi' \leq \varphi_s$$

and so

$$\varphi' \leq \varphi$$

by letting $s \rightarrow 0$.

In order to prove $\varphi \leq \varphi'$, we let

$$v_\delta = \varphi' - \varphi^{(\delta)} - \delta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2.$$

At $t = 0$, $v_\delta = \delta \varphi_0 - \delta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2$.

Suppose $v_\delta(t, z_{min}) = \min_X v_\delta(t, \cdot)$. Then $z_{min} \in X \setminus \tilde{E}$ since both $\varphi^{(\delta)}$ and $\varphi' \in L^\infty(X)$. By the maximum principle, if we choose δ sufficiently small, then at (t, z_{min}) ,

$$\begin{aligned} \frac{\partial}{\partial t} v_\delta &= \log \frac{(\omega_t^{(\delta)} + \sqrt{-1}\partial\bar{\partial}\varphi^{(\delta)} + \delta(\omega_0 - \delta Ric(h_{\tilde{E}})) + \sqrt{-1}\partial\bar{\partial}v_\delta)^n}{(\omega_t^{(\delta)} + \sqrt{-1}\partial\bar{\partial}\varphi^{(\delta)})^n} \\ &\geq \log \frac{(\omega_t^{(\delta)} + \sqrt{-1}\partial\bar{\partial}\varphi^{(\delta)} + \sqrt{-1}\partial\bar{\partial}v_\delta)^n}{(\omega_t^{(\delta)} + \sqrt{-1}\partial\bar{\partial}\varphi^{(\delta)})^n} \\ &\geq 0. \end{aligned}$$

Therefore

$$v_\delta \geq \inf_X v_\delta(0, \cdot) \geq \delta \inf_X (\varphi_0 - \delta \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2),$$

and so we have for $t \in [0, T]$,

$$\varphi^{(\delta)} + \delta \inf_X (\varphi_0 - \delta \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2) + \delta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 \leq \varphi' \leq \varphi.$$

For any $K \subset\subset X \setminus \tilde{E}$, there exists a constant $C_K > 0$ such that

$$|\varphi^{(\delta)} - \varphi| \leq C_K \delta$$

for sufficiently small $\delta > 0$ by Lemma 3.15.

Therefore on K ,

$$\varphi' \geq \varphi + \delta \inf_X (\varphi_0 - \delta \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2) - C_K \delta.$$

Letting $\delta \rightarrow 0$ and then $K \rightarrow X \setminus \tilde{E}$, we have on $X \setminus \tilde{E}$,

$$\varphi' \geq \varphi.$$

Therefore we have proved the uniqueness of the solution on $[0, T] \times X$. The theorem is proved by letting $T \rightarrow T_0$.

□

3.3 Monge-Ampère flows with rough and degenerate initial data

In this section, we will generalize Theorem 3.2 for Monge-Ampère flows with rough initial data. The main result will be applied to the Kähler-Ricci flow on singular projective varieties with surgery.

Let X be an n -dimensional projective manifold. Let L_1 and L_2 be two holomorphic line bundles on X satisfying **Condition A** along with $\omega_0 \in c_1(L_1)$ and $\chi \in c_1(L_2)$ being smooth closed $(1, 1)$ -forms. Let Ω be a non-negative (n, n) -form on X satisfying **Condition B**. Let

$$PSH_p(X, \omega_0, \Omega) = \{\varphi \in PSH(X, \omega_0) \cap L^\infty(X) \mid \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} \in L^p(X, \Omega)\}$$

for $p > 0$ and

$$T_0 = \sup\{t > 0 \mid L_1 + tL_2 \text{ is semi-ample}\}.$$

Since L_1 is big and semi-ample, we denote $Exc(L_1)$ be the exceptional locus for the linear system $|mL_1|$ for sufficiently large m . Without loss of generality, we can assume that \tilde{E} as defined in Section 3.2 contains $Exc(L_1)$.

For any $\varphi_0 \in PSH_p(X, \omega_0, \Omega)$ for some $p > 1$. We define the non-negative function F by the following Monge-Ampère equation

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0)^n = Fe^{\varphi_0}\Theta. \quad (3.29)$$

Lemma 3.16 *Let Θ be a smooth volume form on X and $F = \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0)^n}{\Omega}$. Then there exists $p' > 1$ such that*

$$F \in L^{p'}(X, \Theta). \quad (3.30)$$

Proof The lemma is easily proved by Hölder's inequality and the fact that $\varphi_0 \in L^\infty(X)$.

□

There exist a family of positive functions $\{F_s\}_{s \in (0,1]}$ such that $F_s \in C^\infty(X)$ and

$$\lim_{s \rightarrow 0} \|F_s - F\|_{L^{p'}(X, \Omega)} = 0.$$

We let $F_0 = F$ and then consider the following Monge-Ampère equations

$$(\omega_0 + s\vartheta + \sqrt{-1}\partial\bar{\partial}\varphi_{(0,s)})^n = F_s e^{\varphi_{(0,s)}} \Theta \quad (3.31)$$

and

$$(\omega_0 + s\vartheta + \sqrt{-1}\partial\bar{\partial}\hat{\varphi}_{(s,\gamma)})^n = F_{s+\gamma} e^{\hat{\varphi}_{(s,\gamma)}} \Theta \quad (3.32)$$

Obviously, $\varphi_{(0,s)} \in C^\infty(X)$ and $\hat{\varphi}_{(s,\gamma)} \in C^\infty(X)$ by Yau's theorem [Y2] for $s > 0$ and $\gamma > 0$. Furthermore, both $\|\varphi_{0,s}\|_{L^\infty(X)}$ and $\|\hat{\varphi}_{0,s}\|_{L^\infty(X)}$ are uniformly bounded for $s \in [0, 1]$.

Lemma 3.17 *There exists a decreasing function $\mu(s) \geq 0$ for $s \geq 0$ with $\lim_{s \rightarrow 0} \mu(s) = 0$ such that*

$$\|\hat{\varphi}_{(0,s)} - \varphi_0\|_{L^\infty(X)} \leq \mu(s). \quad (3.33)$$

In particular,

$$\lim_{s \rightarrow 0} \|\hat{\varphi}_{(0,s)} - \varphi_0\|_{L^\infty(X)} = 0. \quad (3.34)$$

Proof Notice F_s converges to F uniformly in $L^{p'}(X, \Omega)$. The lemma follows by combining the proof of the stability theorems for degenerate Monge-Ampère equations in [Ko2] and [DZ] (see Theorem 2.4). \square

Lemma 3.18 *There exists $C > 0$ such that*

$$\hat{\varphi}_{(0,s)} \leq \varphi_{(0,s)} \leq \hat{\varphi}_{(0,s)} + Cs(1 - \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2). \quad (3.35)$$

Proof First let $\psi^+ = \varphi_{(0,s)} - \hat{\varphi}_{(0,s)} - s^m \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2$ for sufficiently large $m > 0$. Notice that $\hat{\varphi}_{0,s} \in C^\infty(X \setminus \tilde{E})$ for $s > 0$. Then

$$\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\hat{\varphi}_{(0,s)} + (s\vartheta - s^m Ric(h_{\tilde{E}})) + \sqrt{-1}\partial\bar{\partial}\psi^+)^n}{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\hat{\varphi}_{(0,s)})^n} = e^{\psi^+ + s^m \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2} \leq e^{\psi^+}.$$

Note that $s\vartheta - s^m Ric(h_{\tilde{E}}) > 0$ for $m \gg 1$. By the maximum principle, $\psi^+ \geq 0$ and so $\hat{\varphi}_{(0,s)} + s^m \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 \leq \varphi_{(0,s)}$. Then by letting $m \rightarrow \infty$, we have

$$\hat{\varphi}_{0,s} \leq \varphi_{0,s}.$$

Now we will bound $\varphi_{(0,s)}$ from above. Let $\psi^- = \frac{1}{1+A^2s}\varphi_{(0,s)} - \hat{\varphi}_{(0,s)} + As \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2$. Then

$$\begin{aligned} & \frac{(1+A^2s)^n(\omega_0 + \sqrt{-1}\partial\bar{\partial}\hat{\varphi}_{(0,s)} - (\frac{A^2s}{1+A^2s}\omega_0 - \frac{s}{1+A^2s}\vartheta - As Ric(h_{\tilde{E}})) + \sqrt{-1}\partial\bar{\partial}\psi^-)^n}{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\hat{\varphi}_{(0,s)})^n} \\ &= e^{\psi^- + \frac{A^2s}{1+A^2s}\varphi_{(0,s)} - As \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2} \\ &\geq e^{\psi^- + \frac{A^2s}{1+A^2s}\varphi_{(0,s)}}. \end{aligned}$$

Note that $\frac{A^2s}{1+A^2s}\omega_0 - \frac{s}{1+A^2s}\vartheta - AsRic(h_{\tilde{E}}) > 0$ for $A \gg 1$ and $s > 0$ sufficiently small. By the maximum principle,

$$\psi^- \leq n \log(1 + A^2s) + \frac{A^2s}{1 + A^2s} \|\varphi_{(0,s)}\|_{L^\infty(X)}.$$

Therefore $\varphi_{(0,s)} \leq \hat{\varphi}_{(0,s)} + Cs - As \log |S_{\tilde{E}}|^2_{h_{\tilde{E}}}$ for some $C > 0$. \square

Corollary 3.3 *Let ϑ be a Kähler form on X . Then for $s > 0$, there exists $\varphi_{0,s} \in PSH(X, \omega_0 + s\vartheta) \cap C^\infty(X)$ such that for any $K \subset\subset X \setminus \tilde{E}$*

$$\lim_{s \rightarrow 0} \|\varphi_{(0,s)} - \varphi_0\|_{L^\infty(K)} = 0. \quad (3.36)$$

Lemma 3.19 *There exist $\gamma_0 > 0$ and $C > 0$ such that for $0 < \gamma < \gamma_0$,*

$$|\varphi_{(0,s+\gamma)} - \varphi_{(0,s)}| \leq C\gamma(1 - \log |S_{\tilde{E}}|^2_{h_{\tilde{E}}}) + C\mu(s + \gamma). \quad (3.37)$$

Proof Notice that

$$|\varphi_{(0,s+\gamma)} - \varphi_{(0,s)}| = |\varphi_{(0,s+\gamma)} - \hat{\varphi}_{(s,\gamma)}| + |\varphi_{(0,s)} - \hat{\varphi}_{(s,\gamma)}|.$$

By the same argument as that in the proof of Lemma 3.18, there exists $C > 0$ such that

$$|\varphi_{(0,s+\gamma)} - \hat{\varphi}_{(s,\gamma)}| \leq Cr(1 - \log |S_{\tilde{E}}|^2_{h_{\tilde{E}}}).$$

Applying the family version of the stability theorem, there exists $C > 0$ such that

$$\|\varphi_{(0,s)} - \hat{\varphi}_{(s,\gamma)}\|_{L^\infty(X)} \leq C\|F_{s+\gamma} - F_s\|_{L^1(X, \Omega)}^{\frac{1}{n+3}}.$$

The Lemma then follows from the above estimates. \square

Consider the following family of Monge-Ampère equations.

$$\begin{cases} \frac{\partial \varphi_{s,w,r}^{(\delta)}}{\partial t} = \log \frac{(\omega_{t,s}^{(\delta)} + \sqrt{-1}\partial\bar{\partial}\varphi_{s,w,r}^{(\delta)})^n}{\Omega_{w,r}}, \\ \varphi_{s,w,r}^{(\delta)}(0, \cdot) = (1 - \delta)\varphi_{(0,s)}, \end{cases} \quad (3.38)$$

where $\omega_{t,s}^{(\delta)} = (1 - \delta)\omega_0 + t\chi + s\vartheta$ is defined as in Section 3.2.

For any $T \in [0, T_0]$, there exists $\delta_0 > 0$ such that for $\delta \in [-\delta_0, \delta_0]$, the equation (3.38) admits a smooth solution on $[0, T] \times X$ as shown in Section 3.2. We will then fix such T and δ_0 .

Lemma 3.20 *There exists $C > 0$ such that for $s, w, r \in (0, 1]$ and $\delta \in [-\delta_0, \delta_0]$,*

$$\|\varphi_{s,w,r}^{(\delta)}(t, \cdot)\|_{L^\infty([0,T] \times X)} \leq C. \quad (3.39)$$

Proof It can be proved by the same argument as that in the proof of Lemma 3.8. \square

Lemma 3.21 *There exists $C > 0$ such that on $[0, T] \times X$, for all $s, w, r \in (0, 1]$ and $\delta \in [-\delta_0, \delta_0]$,*

$$-C \leq t\dot{\varphi}_{s,w,r}^{(\delta)} \leq C. \quad (3.40)$$

Proof The upper bound can be proved using the same argument as that in Lemma 3.2 by applying the maximum principle on

$$H^+ = t\dot{\varphi}_{s,w,r}^{(\delta)} - \varphi_{s,w,r}^{(\delta)}.$$

In order to prove the lower bound. We consider the following family of Monge-Ampère equations

$$(\omega_{0,s} + \sqrt{-1}\partial\bar{\partial}\phi_{s,w,r})^n = A_{s,w,r}\Omega_{w,r}$$

where $A_{s,w,r} = \frac{[\omega_{0,s}]^n}{\int_X \Omega_{w,r}}$ and $\sup_X \phi_{s,w,r} = 0$. Then $A_{s,w,r}$ is uniformly bounded from above and below for s, w and $r \in (0, 1]$. As $A_{s,w,r}\Omega_{w,r}$ is uniformly bounded in $L^p(X, \Theta)$ for some $p > 1$, $\phi_{s,w,r}$ uniformly bounded in $L^\infty(X)$ for $s, w, r \in (0, 1]$.

Let

$$H^- = t\dot{\varphi}_{s,w,r}^{(\delta)} + A^2\varphi_{s,w,r}^{(\delta)} - A\phi_{s^2,w,r}.$$

Let $\Delta_{s,w,r}^{(\delta)}$ be the Laplace operator associated to $\omega_{s,w,r}^{(\delta)}$. Then there exist C_1, C_2 and $C_3 > 0$ such that

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_{s,w,r}^{(\delta)} \right) H^- \\ &= \operatorname{tr}_{\omega_{s,w,r}^{(\delta)}} (A^2\omega_{t,s}^{(\delta)} + t\chi + A\sqrt{-1}\partial\bar{\partial}\phi_{s^2,w,r}) + (A^2 + 1)\dot{\varphi}_{s,w,r}^{(\delta)} - A^2n \\ &\geq \operatorname{tr}_{\omega_{s,w,r}^{(\delta)}} (A\omega_{0,s^2} + A\sqrt{-1}\partial\bar{\partial}\phi_{s^2,w,r}) + (A^2 + 1)\dot{\varphi}_{s,w,r}^{(\delta)} - A^2n \\ &\geq C_1 \left(\frac{(\omega_{0,s^2} + \sqrt{-1}\partial\bar{\partial}\phi_{s^2,w,r})^n}{(\omega_{s,w,r}^{(\delta)})^n} \right)^{\frac{1}{n}} + (A^2 + 1) \log \frac{(\omega_{s,w,r}^{(\delta)})^n}{\Omega_{w,r}} - A^2n \\ &\geq C_2 \left(\frac{\Omega_{w,r}}{(\omega_{s,w,r}^{(\delta)})^n} \right)^{\frac{1}{n}} - C_3 \\ &\geq -C_3 \end{aligned}$$

Applying the maximum principle, H^- is uniformly bounded from below since both $\varphi_{s,w,r}^{(\delta)}$ and $\phi_{s,w,r}$ are uniformly bounded in $L^\infty(X)$. We are done. \square

We have the following volume estimate.

Corollary 3.4 *There exists $C > 0$ such that on $[0, T] \times X$, for all $s, w, r \in (0, 1]$ and $\delta \in [-\delta_0, \delta_0]$,*

$$e^{-\frac{C}{t}} \leq \frac{(\omega_{s,w,r}^{(\delta)})^n}{\Omega_{w,r}} \leq e^{\frac{C}{t}}. \quad (3.41)$$

Lemma 3.22 *There exist $\alpha > 0$ and $C > 0$ such that on $[0, T] \times X$, for all $s, w, r \in (0, 1]$ and $\delta \in [-\delta_0, \delta_0]$,*

$$\operatorname{tr}_\vartheta(\omega_{s,w,r}^{(\delta)}) \leq C|S_{\tilde{E}}|_{h_{\tilde{E}}}^{-\frac{2\alpha}{t}}. \quad (3.42)$$

Proof Define

$$H = t \log \operatorname{tr}_\vartheta(\omega_{s,w,r}^{(\delta)}) - A^2 \varphi_{s,w,r}^{(\delta)} + A \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2.$$

By applying $(\frac{\partial}{\partial t} - \Delta_{s,w,r}^{(\delta)})$ to H , the lemma follows from the maximum principle applied to $(\frac{\partial}{\partial t} - \Delta_{s,w,r}^{(\delta)})H$ and the similar argument in the proof of Lemma 3.3. \square

Lemma 3.23 For $s \in (0, 1]$ and $\delta \in [-\delta_0, \delta_0]$, $\varphi_s^{(\delta)} \in C^\infty((0, T] \times X \setminus \tilde{E})$.

Proof The lemma follows from the standard argument combined with the C^2 -estimate. \square

Obviously for $s, w, r \in (0, 1]$, $\varphi_{s,w,r}^{(\delta)}$ is a smooth family in δ and

$$\lim_{\delta \rightarrow 0} \varphi_{s,w,r}^{(\delta)} = \varphi_{s,w,r}$$

in C^∞ -topology.

For each $s \in (0, 1]$, let

$$\varphi_s^{(\delta)} = \lim_{w \rightarrow 0} (\limsup_{r \rightarrow 0} \varphi_{s,w,r}^{(\delta)})^* \quad (3.43)$$

and

$$\varphi_s = \lim_{w,r \rightarrow 0} \varphi_{s,w,r} = \lim_{w,r \rightarrow 0} \varphi_{s,w,r}^{(0)}. \quad (3.44)$$

Lemma 3.24 There exist constants C and $\delta_0 > 0$ such that for $s, w, r \in (0, 1]$, $t \in [0, T]$ and $\delta \in [-\delta_0, \delta_0]$,

$$C \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - C \leq \frac{\partial}{\partial \delta} \varphi_{s,w,r}^{(\delta)} \leq C. \quad (3.45)$$

Proof The lemma can be proved by the same argument in the proof of Lemma 3.14. \square

Lemma 3.25 There exist $\delta_0 > 0$ and $C > 0$ such that for $s \in [0, 1]$ and $0 < \delta < \delta_0$

$$\varphi_s^{(\delta)} \leq \varphi_{s+\delta^3} - \delta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 + C(\mu(s + \delta) + \delta). \quad (3.46)$$

Proof Let $\psi_\delta = \varphi_{s+\delta^3} - \varphi_s^{(\delta)} - \delta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 + A(\mu(s + \delta) + \delta)$.

Then for each $t \in (0, T]$, at the maximal point of ψ_δ ,

$$\begin{aligned} \frac{\partial}{\partial t} \psi_\delta &= \log \frac{(\omega_s^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi_s^{(\delta)} + \delta(\omega_0 + \delta^2 \vartheta - \delta \operatorname{Ric}(h_{\tilde{E}})) + \sqrt{-1} \partial \bar{\partial} \psi_\delta)^n}{(\omega_s^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi_s^{(\delta)})^n} \\ &\geq \log \frac{(\omega_s^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi_s^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \psi_\delta)^n}{(\omega_s^{(\delta)} + \sqrt{-1} \partial \bar{\partial} \varphi_s^{(\delta)})^n} \\ &\geq 0, \end{aligned}$$

for sufficiently small $\delta > 0$.

For sufficiently large $A > 0$,

$$\psi_\delta|_{t=0} = \varphi_{(0,s+\delta^3)} - (1 - \delta) \varphi_{(0,s)} - \delta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 + A\mu(s + \gamma) + A\delta \geq 0.$$

Therefore $\psi_\delta \leq 0$ on $[0, T] \times X$ by the maximum principle. \square

Lemma 3.26 *There exist $\delta_0 > 0$ and $C > 0$ such that on $[0, T] \times X$, for $s \in [0, 1]$ and $0 < \delta < \delta_0$,*

$$\varphi_{s+\delta^3}^{(\delta)} \leq \varphi_s - \delta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 + C(\mu(s + \delta) + \delta). \quad (3.47)$$

Proof Let $\psi_\delta = \varphi_{s+\delta^3}^{(\delta)} - \varphi_s + \delta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - A(\mu(s + \delta) + \delta)$.

Then for each $t \in (0, T]$, at the minimal point of ψ_δ ,

$$\begin{aligned} \frac{\partial}{\partial t} \psi_\delta &= \frac{(\omega_s + \sqrt{-1}\partial\bar{\partial}\varphi_s - (\delta\omega_0 - \delta^3\vartheta - \delta^2 Ric(h_{\tilde{E}})) + \sqrt{-1}\partial\bar{\partial}\psi_\delta)^n}{(\omega_s + \sqrt{-1}\partial\bar{\partial}\varphi_s)^n} \\ &\leq \frac{(\omega_s + \sqrt{-1}\partial\bar{\partial}\varphi_s + \sqrt{-1}\partial\bar{\partial}\psi_\delta)^n}{(\omega_s + \sqrt{-1}\partial\bar{\partial}\varphi_s)^n} \end{aligned}$$

for sufficiently small $\delta > 0$.

For sufficiently large $A > 0$,

$$\psi_\delta|_{t=0} = (1 - \delta)\varphi_{(0,s+\delta^3)} - \varphi_{(0,s)} + \delta^2 \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - A(\mu(s + \delta) + \delta) \leq 0.$$

Therefore $\psi_\delta \leq 0$ on $[0, T] \times X$ by the maximum principle if $\delta > 0$ is sufficiently small. \square

Then we can show that $\{\varphi_s\}_{s \in (0,1]}$ is a Cauchy family in $L^\infty([0, T] \times K)$ for any compact subset in $X \setminus \tilde{E}$.

Lemma 3.27 *On any $K \subset\subset X \setminus \tilde{E}$,*

$$\lim_{s_1, s_2 \rightarrow 0} \|\varphi_{s_1} - \varphi_{s_2}\|_{L^\infty([0, T] \times K)} = 0. \quad (3.48)$$

Proof Assume $\delta = s_2 - s_1 \geq 0$. Then on $[0, T] \times K$, by Lemma 3.24 and Lemma 3.25, there exist C and $C' > 0$ such that

$$\varphi_{s_1} \leq \varphi_{s_1}^{(\delta^{1/3})} + C\delta^{1/3} \leq \varphi_{s_2} - \delta^{2/3} \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 + C(\mu(s + \delta^{1/3}) + \delta^{1/3})$$

and

$$\begin{aligned} \varphi_{s_1} &\geq \varphi_{s_1+\delta}^{(\delta^{1/3})} + \delta^{2/3} \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - C(\mu(s + \delta^{1/3}) + \delta^{1/3}) \\ &\geq \varphi_{s_2} + \delta^{2/3} \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - C'(\mu(s + \delta^{1/3}) + \delta^{1/3}). \end{aligned}$$

The lemma follows immediately by letting s_1 and $s_2 \rightarrow 0$, $\delta \rightarrow 0$. \square

Proposition 3.5 *For any $\varphi_0 \in PSH_p(X, \omega_0, \Omega)$ for some $p > 1$, there exists a unique $\varphi \in C^\infty((0, T_0) \times (X \setminus \tilde{E}))$ with $\varphi(t, \cdot) \in PSH(X, \omega_0 + t\chi) \cap L^\infty(X)$ for each $t \in (0, T_0)$ such that*

1. $\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_0 + t\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}$ on $(0, T_0) \times X \setminus \tilde{E}$.
2. for any $K \subset\subset X \setminus \tilde{E}$, $\lim_{t \rightarrow 0^-} \|\varphi(t, \cdot) - \varphi_0(\cdot)\|_{L^\infty(K)} = 0$.
3. $\|\varphi\|_{L^\infty((0, T] \times X)}$ is bounded for each $T < T_0$.

Furthermore, for any $T \in (0, T_0)$, there exists $C > 0$ such that on $[0, T] \times X$,

$$e^{-\frac{C}{t}} \leq \frac{(\omega_0 + t\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} \leq e^{\frac{C}{t}}. \quad (3.49)$$

Proof It suffices to show (2) and the volume estimate. On any $K \subset\subset X \setminus \tilde{E}$,

$$\begin{aligned} & \|\varphi(t, \cdot) - \varphi_0(\cdot)\|_{L^\infty(K)} \\ \leq & \|\varphi(t, \cdot) - \varphi_s(\cdot)\|_{L^\infty(K)} + \|\varphi_s(t, \cdot) - \varphi_{(0,s)}(\cdot)\|_{L^\infty(K)} + \|\varphi_{(0,s)}(\cdot) - \varphi_0(\cdot)\|_{L^\infty(K)} \end{aligned}$$

For any $\epsilon > 0$, let s be sufficiently small such that

$$\|\varphi(t, \cdot) - \varphi_s(\cdot)\|_{L^\infty([0, T] \times K)} < \epsilon$$

and

$$\|\varphi_{(0,s)}(\cdot) - \varphi_0(\cdot)\|_{L^\infty(K)} < \epsilon.$$

Fix such s . There exists $t_0 > 0$ such that

$$\|\varphi_s(t, \cdot) - \varphi_{(0,s)}(\cdot)\|_{L^\infty([0, t_0] \times K)} < \epsilon.$$

Therefore $\lim_{t \rightarrow 0^-} \|\varphi(t, \cdot) - \varphi_0(\cdot)\|_{L^\infty(K)} = 0$.

The volume estimate (3.49) follows from (3.41) by letting s, w, r and $\delta \rightarrow 0$. \square

Theorem 3.3 Let X be an n -dimensional algebraic manifold. Let L_1 and L_2 be two holomorphic line bundles on X satisfying **Condition A** and $\omega_0 \in c_1(L_1)$ and $\chi \in c_1(L_2)$ are smooth closed $(1, 1)$ -forms. Let Ω be an (n, n) -form on X satisfying **Condition B**. Let

$$T_0 = \sup\{t > 0 \mid L_1 + tL_2 > 0\}.$$

Then for any $\varphi_0 \in PSH_p(X, \omega_0, \Omega)$ for some $p > 1$, there exists a unique $\varphi \in C^0([0, T_0] \times X \setminus \tilde{E}) \cap C^\infty((0, T_0) \times (X \setminus \tilde{E}))$ with $\varphi(t, \cdot) \in PSH(X, \omega_0 + t\chi) \cap L^\infty(X)$ for each $t \in [0, T_0)$ such that

1. $\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_0 + t\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}$ on $(0, T_0) \times X \setminus \tilde{E}$,
2. $\varphi|_{t=0} = \varphi_0$ on X .

Furthermore, for any $T \in (0, T_0)$, there exists $C > 0$ such that on $[0, T] \times X$,

$$e^{-\frac{C}{t}} \leq \frac{(\omega_0 + t\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} \leq e^{\frac{C}{t}}. \quad (3.50)$$

Proof It suffices to show the uniqueness of the solution φ and it can be proved by the similar argument for the proof of Theorem 3.2. \square

4 Kähler-Ricci flow on varieties with log terminal singularities

4.1 Notations

Let X be a \mathbf{Q} -factorial projective variety with at worst log terminal singularities. We denote the singular set of X by X_{sing} and let $X_{reg} = X \setminus X_{sing}$. Let $\pi : \tilde{X} \rightarrow X$ be the resolution of singularities and $K_{\tilde{X}} = \pi^* K_X + \sum a_i E_i$ where E_i is the irreducible component of the exceptional locus $Exc(\pi)$ of π . Since X is log terminal, $a_i > -1$.

Since K_X is a \mathbf{Q} -Cartier divisor, there exists a positive $m \in \mathbf{Z}$ such that mK_X is Cartier.

Definition 4.1 Ω is said to be a smooth volume form on X if Ω is a smooth (n, n) -form on X such that for any $z \in X$, there exists an open neighborhood U of z such that

$$\Omega = f_U(\alpha \wedge \bar{\alpha})^{\frac{1}{m}},$$

where f_U is a smooth positive function on U and α is a local generator of mK_X on U .

On each U , $\sqrt{-1}\partial\bar{\partial}\log(\alpha \wedge \bar{\alpha}) = 0$ on U and $\sqrt{-1}\partial\bar{\partial}\log f_U$ is a well-defined smooth closed $(1, 1)$ -form on U if we extend f_U in the ambient space of U . Then $\chi = \sqrt{-1}\partial\bar{\partial}\log \Omega$ is a well-defined smooth closed $(1, 1)$ -form on X , furthermore, $\chi \in [K_X]$. We can then define $Ric(\Omega)$ to be $\chi = \sqrt{-1}\partial\bar{\partial}\log h_\Omega$, where $h_\Omega = \Omega^{-1}$ defines a smooth hermitian metric on K_X .

After pulling back Ω by the resolution π , $\pi^*\Omega$ is then a non-negative (n, n) -form on \tilde{X} . In particular, $\pi^*\Omega$ has zeros or poles along the exceptional divisor E_i of order $|a_i|$ and

$$\pi^*\chi = Ric(\pi^*h_\Omega).$$

Let D be an ample divisor on \tilde{X} such that

$$\omega_D = Ric(h_D) = -\sqrt{-1}\partial\bar{\partial}\log h_D > 0$$

where h_D is a hermitian metric equipped on the line bundle associated to D .

Let $\iota : X \rightarrow \mathbf{CP}^N$ be any imbedding of X into a projective space and ω_0 be the pullback of a smooth Kähler metric from \mathbf{CP}^N in a multiple of the Kähler class $\mathcal{O}(1)$. Then ω_0 is a smooth Kähler metric on X . Since $[\pi^*\omega_0]$ is the pullback of an ample class on \mathbf{CP}^N , it is a big and semi-ample divisor on \tilde{X} . By the Kodaira's lemma, there exists an effective divisor \tilde{E} on \tilde{X} such that

$$[\pi^*\omega_0] - \epsilon[\tilde{E}]$$

is ample for any $\epsilon > 0$ sufficiently small. Furthermore, since X is \mathbf{Q} -factorial, we can assume by Proposition 2.1 that the support of \tilde{E} is contained in the exceptional locus of π . There exists a hermitian metric $h_{\tilde{E}}$ equipped on the line bundle associated to \tilde{E} such that for sufficiently small $\epsilon > 0$,

$$\pi^*\omega_0 - \epsilon Ric(h_{\tilde{E}}) > 0.$$

Let S_D and $S_{\tilde{E}}$ be the defining section of D and \tilde{E} .

Definition 4.2 Let X be a \mathbf{Q} -factorial projective variety with log terminal singularities and H be a big and semi-ample \mathbf{Q} -divisor on X . Let $\omega_0 \in [H]$ be a smooth closed $(1, 1)$ -form and Ω a smooth volume form on X . We define for $p \in (0, \infty]$,

$$PSH_p(X, \omega_0, \Omega) = \{\varphi \in PSH(X, \omega_0) \cap L^\infty(X) \mid \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} \in L^p(X, \Omega)\}. \quad (4.1)$$

and

$$\mathcal{K}_{H,p}(X) = \{\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi \mid \varphi \in PSH_p(X, \omega_0, \Omega)\}. \quad (4.2)$$

The definition of $\mathcal{K}_{H,p}(X)$ does not depend on the choice of the smooth closed $(1,1)$ -form $\omega_0 \in [H]$ and the smooth volume form Ω .

We define the following notion of the weak Kähler-Ricci flow on projective varieties with singularities.

Definition 4.3 *Let X be a \mathbf{Q} -factorial projective variety with log terminal singularities and $\omega_0 \in [H]$ be a closed semi-positive $(1,1)$ -current on X associated to a big and semi-ample \mathbf{Q} -divisor H on X . Suppose that*

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\} > 0.$$

A family of closed positive $(1,1)$ -current $\omega(t, \cdot)$ on X for $t \in [0, T_0)$ is called a solution of the unnormalized weak Kähler-Ricci flow if the following conditions hold.

1. $\omega \in C^\infty((0, T_0) \times X \setminus D)$, where D is a subvariety of X . Let $\hat{\omega}_t \in [H + tK_X]$ be a smooth family of smooth closed $(1,1)$ -forms on X for $t \in [0, T_0)$. Then $\omega = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$ for some $\varphi \in C^0([0, T_0) \times X \setminus D) \cap C^\infty((0, T_0) \times X \setminus D)$ and $\varphi(t, \cdot) \in PSH(X, \hat{\omega}_t) \cap L^\infty(X)$ for all $t \in [0, T_0)$.

2.

$$\begin{cases} \frac{\partial \omega}{\partial t} = -Ric(\omega), & \text{on } (0, T_0) \times X \setminus D, \\ \omega(0, \cdot) = \omega_0, & \text{on } X. \end{cases} \quad (4.3)$$

In particular, when H is ample, T_0 is always positive and $X \setminus D = X_{reg}$.

We would like to prove the existence and uniqueness of the weak Kähler-Ricci flow on singular varieties if the initial metric satisfies certain regularity conditions.

The following theorems are well-known as the rationality theorem and base-point-free theorem in the Minimal Model Program (see [KMM], [D]).

Theorem 4.1 *Let X be a projective manifold such that K_X is not nef. Let H be an ample \mathbf{Q} -divisor and let*

$$\lambda = \max\{t \in \mathbf{R} \mid H + tK_X \text{ is nef}\}. \quad (4.4)$$

Then $\lambda \in \mathbf{Q}$.

Theorem 4.2 *Let X be a projective manifold. Let D be a nef \mathbf{Q} -divisor such that $aD - K_X$ is nef and big for some $a > 0$. Then D is semi-ample.*

4.2 Existence and uniqueness of the weak Kähler-Ricci flow

Let X be a \mathbf{Q} -factorial projective variety with log terminal singularities. Let H be an ample \mathbf{Q} -divisor on X , $\omega_0 \in [H]$ be a smooth Kähler metric and Ω be a smooth volume form on X and $\chi = \sqrt{-1}\partial\bar{\partial}\log\Omega$.

Consider the ordinary differential equation for the Kähler class defined by the unnormalized Ricci flow on X

$$\left\{ \begin{array}{l} \frac{\partial[\omega]}{\partial t} = [\chi] = [K_X], \\ [\omega(t, \cdot)] = [\omega_0] = [H]. \end{array} \right. \quad (4.5)$$

Then

$$[\omega] = [\omega_0] + t[\chi].$$

Heuristically, if the Kähler-Ricci flow exists for $t \in [0, T)$, the unnormalized Kähler-Ricci flow should be equivalently to the following Monge-Ampère flow

$$\left\{ \begin{array}{l} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_0 + t\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} \\ \varphi(0, \cdot) = 0. \end{array} \right. \quad (4.6)$$

Let $\pi : \tilde{X} \rightarrow X$ be the resolution of singularities as defined in Section 4.1. In order to define the Monge-Ampère flow on X , one might want to lift the flow to the nonsingular model \tilde{X} of X . However, ω_0 is not necessarily Kähler on \tilde{X} and Ω in general vanishes or blows up along the exceptional divisor of π on \tilde{X} unless the resolution π is crepant, hence the lifted flow is degenerate near the exceptional locus. So we have to perturb the Monge-Ampère flow (4.6) and obtain uniform estimates so that the flow might be allowed to be pushed down on X .

Let

$$T_0 = \sup\{t \geq 0 \mid [\omega_t] \text{ is nef on } X\} = \sup\{t \geq 0 \mid H + tK_X \text{ is nef on } X\}.$$

Then for any $t \in [0, T_0)$, $[\omega_t]$ is ample and $T_0 > 0$ is a rational number or $T_0 = \infty$ by the rationality theorem 4.1. The base-point-free theorem 4.2 implies the following important proposition.

Proposition 4.1 *When $T_0 < \infty$, $H + T_0K_X$ is semi-ample.*

Theorem 4.3 *Let $\varphi_0 \in PSH_p(X, \omega_0, \Omega)$ for some $p > 1$. Then the Monge-Ampère flow on $\tilde{X} \setminus \tilde{E}$*

$$\left\{ \begin{array}{l} \frac{\partial \tilde{\varphi}(t, \cdot)}{\partial t} = \log \frac{(\pi^*\omega_t + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi})^n}{\pi^*\Omega} \\ \tilde{\varphi}(0, \cdot) = \pi^*\varphi_0 \end{array} \right. \quad (4.7)$$

has a unique solution $\tilde{\varphi} \in C^\infty((0, T_0) \times \tilde{X} \setminus \tilde{E}) \cap C^0([0, T_0) \times \tilde{X} \setminus \tilde{E})$ such that for all $t \in [0, T_0)$, $\varphi(t, \cdot) \in L^\infty(\tilde{X}) \cap PSH(\tilde{X}, \pi^\omega_t)$. Furthermore, $\tilde{\varphi}$ is constant along each connected fibre of π , hence $\tilde{\varphi}$ descends to a unique solution $\varphi \in C^\infty((0, T_0) \times X_{reg}) \cap C^0([0, T_0) \times X_{reg})$ of the Monge-Ampère flow (4.6) such that for each $t \in [0, T_0)$, $\varphi \in PSH(X, \omega_t) \cap L^\infty(X)$.*

Proof Since $[\pi^*\omega_0]$ corresponds to a big and semi-ample divisor on \tilde{X} and $[\pi^*\omega_0] - \epsilon[\pi^*\chi]$ is also big and semi-ample for sufficiently small $\epsilon > 0$. The adjunction formula gives $K_{\tilde{X}} = \pi^*K_X + \sum_i a_i E_i + \sum_j F_j$, where E_i and F_j are irreducible components of the exceptional locus with $a_i \geq 0$ and $b_j > -1$. Note that $\pi^*\Omega$ vanishes only on each E_i to order a_i and $\pi^*\Omega$ has poles along those F_j with b_j . Then $\pi^*\omega_0$, $\pi^*\chi$ and $\pi^*\Omega$ satisfy **Condition A** and **Condition B**. Furthermore, $\pi^*\varphi_0 \in PSH_p(\tilde{X}, \pi^*\omega_0, \pi^*\Omega)$ and so the assumptions in Theorem 3.3 are satisfied. The first part of the theorem is then an immediate corollary of Theorem 3.3.

The singular set \tilde{E} can be chosen to be contained in the exceptional locus $Exc(\pi)$ of π , since X is \mathbf{Q} -factorial. Also $\tilde{\varphi}$ must be constant along each component of $Exc(\pi)$ as $[\pi^*\omega_t]$ is trivial along each component of the exceptional divisors. So it descends to a function in $PSH(X, \omega_t)$ on X . \square

Theorem 4.4 *Let X be a \mathbf{Q} -factorial projective variety with log terminal singularities and H be an ample \mathbf{Q} -divisor on X . Let*

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}.$$

If $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$, then there exists a unique solution ω of the unnormalized weak Kähler-Ricci flow for $t \in [0, T_0]$.

Furthermore, if Ω is a smooth volume form on X , then for any $T \in (0, T_0)$, there exists a constant $C > 0$ such that on $[0, T] \times X$,

$$e^{-\frac{C}{t}}\Omega \leq \omega^n \leq e^{\frac{C}{t}}\Omega. \quad (4.8)$$

Proof It suffices to prove the uniqueness as the existence and the volume estimate follow easily from Theorem 4.3 and Theorem 3.3.

Let $\omega_t = \omega_0 + t\chi$ and then $\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi$ with $\varphi \in L^\infty(\tilde{X}) \cap C^\infty(\tilde{X} \setminus \tilde{E})$. Then the Kähler-Ricci flow is equivalent to the following equation

$$\begin{cases} \sqrt{-1}\partial\bar{\partial}\left(\frac{\partial}{\partial t}\varphi - \log\frac{\omega^n}{\Omega}\right) = 0, & \text{on } \tilde{X} \setminus \tilde{E} \\ \varphi(0, \cdot) = 0. \end{cases} \quad (4.9)$$

Let $F = \frac{\partial}{\partial t}\varphi - \log\frac{\omega^n}{\Omega}$. Then $F \in C^\infty(\tilde{X} \setminus \tilde{E})$ and $\sqrt{-1}\partial\bar{\partial}F = 0$ on $\tilde{X} \setminus \tilde{E}$.

Since X is \mathbf{Q} -factorial, $\pi^*[\omega_0] - \epsilon[Exc(\pi)]$ is ample for $\epsilon > 0$ sufficiently small. So we can choose \tilde{E} to be contained in $Exc(\pi)$. Hence F descends to X_{reg} and $\sqrt{-1}\partial\bar{\partial}F = 0$ on X_{reg} . For each $t \in (0, T_0)$, F is smooth on X_{reg} , therefore F is constant on each curve in X which does not intersect X_{sing} . On the other hand, for any two generic points z and w on X , there exists a curve joining z and w without intersecting X_{sing} since $codim(X_{sing}) \geq 2$. So $F(z) = F(w)$ as F is constant on C . Then F is constant on X_{reg} since F is continuous on X_{reg} .

By modifying φ by a function only in t , φ would satisfy the Monge-Ampère flow (4.7). The theorem follows from the uniqueness of the solution φ .

The volume estimate also follows from Theorem 3.3. \square

We immediately have the following long time existence result generalizing the case for non-singular minimal models due to Tian-Zhang [TiZha].

Corollary 4.1 *Let X be a minimal model with log terminal singularities and H be an ample \mathbf{Q} -divisor on X . Then*

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\} = \infty$$

and the unnormalized weak Kähler-Ricci flow starting with $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$ exists for $t \in [0, \infty)$.

4.3 Kähler-Ricci flow on projective varieties with orbifold or crepant singularities

Given a normal projective variety X , very little is known how to construct "good" Kähler metrics on X with reasonable curvature conditions. In general, the restriction of Fubini-Study metrics ω_{FS} on X from ambient projective spaces behaves badly near the singularities of X . Even the scalar curvature of ω_{FS} would have to blow up. In particular, ω^n is not necessarily equivalent to a smooth volume form on X . For example, let X be a surface containing a curve C with self-intersection number -2 and Y be the surface obtained from X by contracting C . Then Y has an isolated orbifold singularity. Let ω be a smooth Kähler metric and Ω a smooth volume on Y . Then $\frac{\omega^n}{\Omega} = 0$ at the orbifold singularity. It tells that one should look at the category of smooth orbifold Kähler metrics on Y instead of smooth Kähler metrics from ambient spaces.

As it turns out, the Kähler-Ricci flow produces Kähler currents whose Monge-Ampère mass is equivalent to a smooth volume form on singular varieties by Theorem 4.4. It is desirable that the Kähler-Ricci flow indeed improves the regularity of the initial data. In the case when X has orbifold or crepant singularities, we show that at least the scalar curvature of the Kähler currents are bounded. In particular if X has only orbifold singularities, the Kähler-Ricci flow immediately smoothes out the initial Kähler current.

Theorem 4.5 *Let X be a \mathbf{Q} -factorial projective normal variety with orbifold singularities. Let H be an ample \mathbf{Q} -divisor on X and*

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}.$$

If $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$, then there exists a unique solution ω of the unnormalized weak Kähler-Ricci flow for $t \in [0, T_0)$.

Furthermore, $\omega(t, \cdot)$ is a smooth orbifold Kähler-metric on X for all $t > 0$ and so the weak Kähler-Ricci flow becomes the smooth Kähler-Ricci flow on X immediately when $t > 0$.

Proof X is automatically log terminal under the assumptions in the theorem if it only admits orbifold singularities. The theorem can be proved by the same argument as in Theorem 3.1. We leave the details for the readers as an exercise. □

Theorem 4.5 can also be applied to the Kähler-Ricci flow on projective manifolds whose initial class is not Kähler.

Theorem 4.6 *Let X be a smooth projective variety. Let H be a big and semi-ample \mathbf{Q} -divisor on X . Suppose that*

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is semi-ample}\} > 0.$$

If $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$, then there exists a unique solution ω of the unnormalized weak Kähler-Ricci flow for $t \in [0, T_0)$.

Furthermore, for any $t \in (0, T_0)$, there exists $C(t) > 0$ such that the scalar curvature $S(\omega(t, \cdot))$ is bounded by $C(t)$

$$\|S(\omega(t, \cdot))\|_{L^\infty(X)} \leq C(t). \tag{4.10}$$

Proof Let Ω be a smooth volume form on X and $\chi = \sqrt{-1}\partial\bar{\partial}\log\Omega$. Let ϑ be a Kähler form on X . Suppose that $\omega_0 = \omega'_0 + \sqrt{-1}\partial\bar{\partial}\phi$, where $\omega'_0 \in [H]$ is a smooth Kähler current and $\phi \in PSH_p(X, \omega'_0)$ for some $p > 1$. We consider the special case of the Monge-Ampere flow (3.38) by letting $\delta = w = r = 0$ and $\omega'_{t,s} = \omega'_0 + s\vartheta + t\chi$.

$$\frac{\partial\varphi_s}{\partial t} = \log \frac{(\omega'_{t,s} + \sqrt{-1}\partial\bar{\partial}\varphi_s)^n}{\Omega}, \quad \varphi_s|_{t=0} = \varphi_{(0,s)}. \quad (4.11)$$

In fact, equation (4.11) is equivalent to the unnormalized Kähler-Ricci flow on X starting with ω_0 . Furthermore, $\varphi_s(t, \cdot)$ is smooth for $t \in (0, T_0)$. Let $\tilde{\omega}_s(t, \cdot) = \omega'_{t,s} + \sqrt{-1}\partial\bar{\partial}\varphi_s$. Then for $t > 0$,

$$\frac{\partial\tilde{\omega}_s}{\partial t} = -Ric(\tilde{\omega}_s)$$

and so

$$\frac{\partial}{\partial t}S(\tilde{\omega}_s) = \Delta_s S(\tilde{\omega}_s) + |Ric(\tilde{\omega}_s)|^2,$$

where Δ_s is the Laplace operator associated to $\tilde{\omega}_s$.

Since

$$(\frac{\partial}{\partial t} - \Delta_s)tS(\tilde{\omega}_s) = S(\tilde{\omega}_s) + t|Ric(\tilde{\omega}_s)|^2 \geq S(\tilde{\omega}_s) + \frac{t}{n}S(\tilde{\omega}_s)^2.$$

The maximum principle immediately implies that $tS(\tilde{\omega}_s)$ is bounded from below on $[0, T_0] \times X$ uniformly in $s \in (0, 1]$. By letting $s \rightarrow 0$, $tS(\omega(t, \cdot))$ is uniformly bounded from below on $[0, T_0] \times X$.

Now we will prove the upper bound for $S(\omega)$.

Claim 1 For any $0 < t_0 < T < T_0$, there exist A and $B > 0$ such that for all $s \in (0, 1]$ and on $[t_0, T] \times X$,

$$(\frac{\partial}{\partial t} - \Delta_s)tr_{\tilde{\omega}_s}(\omega'_0) \leq A(tr_{\tilde{\omega}_s}(\omega'_0))^2 - B|\nabla_s tr_{\tilde{\omega}_s}(\omega'_0)|^2, \quad (4.12)$$

$$(\frac{\partial}{\partial t} - \Delta_s)tr_{\tilde{\omega}_s}(\omega'_0 + T\chi) \leq A(tr_{\tilde{\omega}_s}(\omega'_0 + T\chi))^2 - B|\nabla_s tr_{\tilde{\omega}_s}(\omega'_0 + T\chi)|^2, \quad (4.13)$$

where ∇_s is the gradient operator associated to $\tilde{\omega}_s$.

Claim 2 For any $0 < t_0 < T < T_0$, There exists $C > 0$ such that for all $s \in (0, 1]$ and on $[t_0, T] \times X$,

$$0 \leq tr_{\tilde{\omega}_s}(\omega'_{t,0}) < C. \quad (4.14)$$

In particular, there exists $C > 0$ such that

$$0 \leq tr_{\tilde{\omega}_s}(\omega'_0) < C, \quad -C < tr_{\tilde{\omega}_s}(\chi) < C. \quad (4.15)$$

Straightforward calculations show that

$$\begin{aligned} & (\frac{\partial}{\partial t} - \Delta_s)(|\nabla_s \frac{\partial\varphi_s}{\partial t}|^2) \\ &= -|\nabla_s \nabla_s \frac{\partial\varphi_s}{\partial t}|^2 - |\bar{\nabla}_s \nabla_s \frac{\partial\varphi_s}{\partial t}|^2 + (\nabla_s tr_{\tilde{\omega}_s}(\chi) \cdot \bar{\nabla}_s \frac{\partial\varphi_s}{\partial t} + \nabla_s \frac{\partial\varphi_s}{\partial t} \cdot \bar{\nabla}_s tr_{\tilde{\omega}_s}(\chi)) \end{aligned} \quad (4.16)$$

and

$$\left(\frac{\partial}{\partial t} - \Delta_s\right)\Delta_s \frac{\partial \varphi_s}{\partial t} = -|\bar{\nabla}_s \nabla_s \frac{\partial \varphi_s}{\partial t}|^2 - g^{i\bar{l}} g^{k\bar{j}} \chi_{i\bar{j}} \left(\frac{\partial \varphi_s}{\partial t}\right)_{k\bar{l}} \quad (4.17)$$

Notice that $\frac{\partial \varphi_s}{\partial t}$ is uniformly bounded on $[t_0, T]$ for any $0 < t_0 < T < T_0$ and $s \in (0, 1]$.

Then similar argument in the proof of Theorem 5.1 can be applied. Namely, one can apply the maximum principle for $(t - t_0)\mathcal{H}$ and $(t_0)\mathcal{K}$, where

$$\mathcal{H} = \frac{|\nabla_s \frac{\partial \varphi_s}{\partial t}|^2}{A - \frac{\partial \varphi_s}{\partial t}} + \text{tr}_{\tilde{\omega}_s}(\omega'_0) + \text{tr}_{\tilde{\omega}_s}(\omega'_0 + T\chi)$$

and

$$\mathcal{K} = -\frac{\Delta_s \frac{\partial \varphi_s}{\partial t}}{A - u} + B\mathcal{H}.$$

If we choose $A > 0$ sufficiently large,

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_s\right)(t - t_0)\mathcal{H} \\ & \leq -\epsilon(t - t_0) \frac{|\frac{\partial \varphi_s}{\partial t}|^4}{(A - \frac{\partial \varphi_s}{\partial t})^3} - \frac{2(1 - \epsilon)(t - t_0)}{A - u} \text{Re}(\nabla_s \mathcal{H} \cdot \bar{\nabla}_s \frac{\partial \varphi_s}{\partial t}) + C_1\mathcal{H} + C_1 \\ & \leq -\epsilon C_2(t - t_0)\mathcal{H}^2 + C_1\mathcal{H} - \frac{2(1 - \epsilon)(t - t_0)}{A - u} \text{Re}(\nabla_s \mathcal{H} \cdot \bar{\nabla}_s \frac{\partial \varphi_s}{\partial t}) + C_3. \end{aligned}$$

Hence $(t - t_0)\mathcal{H}$ is uniformly bounded on $[t_0, T] \times X$ for any $s \in (0, 1]$.

If we choose A and $B > 0$ sufficiently large,

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_s\right)(t - t_0)\mathcal{K} \\ & \leq -C_4(t - t_0) \frac{|\nabla_s \bar{\nabla}_s \frac{\partial \varphi_s}{\partial t}|^2}{A - \frac{\partial \varphi_s}{\partial t}} - \frac{2(t - t_0)}{A - u} \text{Re}(\nabla_s \mathcal{K} \cdot \bar{\nabla}_s \frac{\partial \varphi_s}{\partial t}) + C_1\mathcal{K} + C_1 \\ & \leq -C_5(t - t_0)\mathcal{K}^2 + C_1\mathcal{K} - \frac{2(t - t_0)}{A - u} \text{Re}(\nabla_s \mathcal{K} \cdot \bar{\nabla}_s \frac{\partial \varphi_s}{\partial t}) + C_6. \end{aligned}$$

Hence $(t - t_0)\mathcal{K}$ is uniformly bounded on $[t_0, T] \times X$ for any $s \in (0, 1]$. Here we make use of Claim 1 that

$$T \text{tr}_{\tilde{\omega}_s}(\chi) = \text{tr}_{\tilde{\omega}_s}(\omega'_0 + T\chi) - \text{tr}_{\tilde{\omega}_s}(\omega'_0)$$

is uniformly bounded on $[t_0, T] \times X$ uniformly for $s \in (0, 1]$. Also the term

$$T^2 |\nabla_s \text{tr}_{\tilde{\omega}_s}(\chi)|^2 \leq |\nabla_s \text{tr}_{\tilde{\omega}_s}(\omega'_0 + T\chi)|^2 + |\nabla_s \text{tr}_{\tilde{\omega}_s}(\omega'_0)|^2$$

can be controlled by $(\frac{\partial}{\partial t} - \Delta_s)(\text{tr}_{\tilde{\omega}_s}(\omega'_0) + \text{tr}_{\tilde{\omega}_s}(\omega'_0 + T\chi))$.

Therefore there exists $C > 0$ such that on $[t_0, T] \times X$

$$S(\tilde{\omega}_s) = -\Delta_s \frac{\partial \varphi_s}{\partial t} - \text{tr}_{\tilde{\omega}_s}(\chi) \leq C$$

uniformly in $s \in (0, 1]$.

The theorem is then proved by letting $s \rightarrow 0$.

□

Now we shall prove the two claims in the proof of Theorem 4.6.

Proof of Claim 1 Without loss of generality, we let $\pi : X \rightarrow \mathbf{CP}^{N_m}$ be the morphism induced by mH and $m\omega'_0$ is the pullback of the Fubini-Study metric on \mathbf{CP}^{N_m} if m is sufficiently large. Notice that for $t \in [t_0, T]$, $H + tK_X$ is still semi-ample and big, so $m'(H + tK)$ induces a morphism $\pi' : X \rightarrow \mathbf{CP}^{N_{m'}}$. We can again assume that $\omega_0 + t\chi$ is the pullback of the Fubini-Study metric on $\mathbf{CP}^{N_{m'}}$. The curvature of ω'_0 on \mathbf{CP}^{N_m} and the curvature of $\omega'_0 + t\chi$ on $\mathbf{CP}^{N_{m'}}$ are both bounded. Then it becomes a straightforward calculation from [SoT1]. \square

Proof of Claim 2 This can be proved by the parabolic Schwarz lemma from [SoT1]. We apply the maximum principle for $t \log \text{tr}_{\tilde{\omega}_s}(\omega'_0) - A\varphi_s$ and $t \log \text{tr}_{\tilde{\omega}_s}(\omega'_0 + t\chi) - A\varphi_s$ for sufficiently large A so that both terms are uniformly bounded on $[0, T] \times X$ uniformly for $s \in (0, 1]$. The claim then easily follows. \square

Theorem 4.6 shows that the Kähler-Ricci flow can be defined even if the initial smooth Kähler current is in the semi-ample cone of divisors.

The following theorem is an immediate corollary of Theorem 4.6.

Theorem 4.7 *Let X be a \mathbf{Q} -factorial projective variety with crepant singularities. Let H be an ample \mathbf{Q} -divisor on X and*

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}.$$

If $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$, then there exists a unique solution ω of the unnormalized weak Kähler-Ricci flow for $t \in [0, T_0]$.

Furthermore, for any $t \in (0, T_0)$, there exists $C(t) > 0$ such that the scalar curvature $S(\omega(t, \cdot))$ is bounded by $C(t)$

$$\|S(\omega(t, \cdot))\|_{L^\infty(X)} \leq C(t). \quad (4.18)$$

Proof Let Ω be a smooth volume form on X . Let $\pi : \tilde{X} \rightarrow X$ be a crepant resolution of X . Then $\pi^*\Omega$ is again a smooth volume form on \tilde{X} . Then we can apply Theorem 4.6.

We remark that it might be interesting to remove the dependence on the t for the scalar curvature bound if $T_0 = \infty$. \square

It is shown in [Z2] that the scalar curvature is uniformly bounded along the normalized Kähler-Ricci flow on smooth manifolds of general type. On the other hand, the scalar will in general blow up if the Kähler-Ricci flow develops finite time singularities (see [Z3]).

5 Kähler-Ricci flow with surgery

5.1 Minimal Model Program with Scaling

Definition 5.1 *Let X be a projective variety and $N_1(X)_{\mathbf{Z}}$ the group of numerically equivalent 1-cycles (two 1-cycles are numerically equivalent if they have the same intersection number with every Cartier divisor). Let $N_1(X)_{\mathbf{R}} = N_1(X)_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{R}$. We denote by $NE(X)$ the set of classes of effective 1-cycles. $NE(X)$ is convex and we let $\overline{NE}(X)$ be the closure of $NE(X)$ in the Euclidean topology.*

A special case of the Minimal Model Program is proposed in [BCHM] and plays an important role for the termination of flips. We briefly explain the Minimal Model Program with Scaling below.

Definition 5.2 (MMP with scaling)

1. We start with a pair (X, H) , where X is a projective \mathbf{Q} -factorial variety X with log terminal singularities and H is a big and semi-ample \mathbf{Q} -divisor on X .
2. Let $\lambda_0 = \inf\{\lambda > 0 \mid \lambda H + K_X \text{ is nef}\}$ be the nef threshold. If $\lambda_0 = 0$, then we stop since K_X is already nef.
3. Otherwise, there is an extremal ray R of the cone of curves $\overline{NE}(X)$ on which K_X is negative and $\lambda_0 H + K_X$ is zero. So there exists a contraction $\pi : X \rightarrow Y$ of R .
 - If π is a divisorial contraction, we replace X by Y and let H_Y be the strict transformation of $\lambda_0 H + K_X$ by π . Then we return to 1. with (Y, H_Y) .
 - If π is a small contraction, we replace X by its flip X^+ and let H_{X^+} be the strict transformation of $\lambda_0 H + K_X$ by the flip. Then we return to 1. with (X^+, H_{X^+}) .
 - If $\dim Y < \dim X$, then X is a Mori fibre space, i.e., the fibers of π are Fano. Then we stop.

The following theorem is proved in [BCHM].

Theorem 5.1 *If X is of general type, the Minimal Model Program with Scaling terminates in finite steps.*

In general, the contraction of the extremal ray might not be the same as the contraction induced by the semi-ample divisor $\lambda_0 H + K_X$. We define the following special ample divisors so that at each step, there is only one extremal ray contracted by the morphism induced by $\lambda_0 H + K_X$.

Definition 5.3 *Let X be a projective \mathbf{Q} -factorial variety with log terminal singularities. An ample \mathbf{Q} -divisor H on X is called a good initial divisor H if the following conditions are satisfied.*

1. Let $X_0 = X$ and $H_0 = H$. The MMP with scaling terminates in finite steps by replacing (X_0, H_0) by (X_1, H_1) , ..., (X_m, H_m) until X_{m+1} is a minimal model or X_m is a Mori fibre space.
2. Let λ_i be the nef threshold for each pair (X_i, H_i) for $i = 1, \dots, m$. Then the contraction induced by the semi-ample divisor $\lambda_i H_i + K_{X_i}$ contracts exactly one extremal ray.

It might be possible that good initial divisors are generic if MMP with scaling holds for any pair (X, H) . It will be seen in the future that good initial divisors simplify the analysis for surgery along the Kähler-Ricci flow, though such an assumption is not necessary. We will explain it in detail in Section 5.5.

Now we relate the Kähler-Ricci flow to MMP with scaling. Consider the unnormalized Kähler-Ricci flow $\frac{\partial \omega}{\partial t} = -Ric(\omega)$ on X with the initial Kähler current $\omega_0 \in [H]$ for an ample divisor H on X . Let $T_0 = \sup\{t \geq 0 \mid H + tK_X > 0\}$. By the rationality theorem 4.1, $T_0 = \infty$ or T_0 is a positive rational number. In particular, if X is a minimal model, then $T_0 = \infty$. In fact,

$T_0 = \frac{1}{\lambda_0}$ is the inverse of the nef threshold. The following theorem is a natural generalization for the long time existence theorem of Tian and Zhang [TiZha] for the Kähler-Ricci flow on smooth minimal models.

Theorem 5.2 *Let X be an n -dimensional \mathbf{Q} -factorial projective variety with log terminal singularities with nef K_X . For any ample \mathbf{Q} -divisor H on X and $\omega_0 \in \mathcal{K}_{H,p}(X)$ with $p > 1$, there unnormalized weak Kähler-Ricci flow starting with ω_0 , exists for $t \in [0, \infty)$.*

Suppose that X is not minimal and so $T_0 < \infty$. Then $H + T_0 K_X$ is nef and the weak Kähler-Ricci exists uniquely for $t \in [0, T_0)$. By Kawamata's base point free theorem, $H + T_0 K_X$ is semi-ample and hence the ring $R(X, H + T_0 K_X) = \bigoplus_{m=0}^{\infty} H^0(X, m(H + T_0 K_X))$ is finitely generated.

- If $H + T_0 K_X$ is big and hence $R(X, H + T_0 K_X)$ induces a birational morphism $\pi : X \rightarrow Y$. For a generic ample divisor H , the morphism π contracts exactly one extremal ray of $\overline{NE}(X)$. We discuss the following two cases according to the size of the exceptional locus of π .

1. π is a divisorial contraction, that is, the exceptional locus $Exc(\pi)$ is a divisor whose image of π has codimension at least two. In this case, Y is still \mathbf{Q} -factorial and has at worst log terminal singularities.
2. π is a small contraction, that is, the exceptional locus $Exc(\pi)$ has codimension at least two. In this case, Y have rather bad singularities and K_Y is no longer a Cartier \mathbf{Q} -divisor. The solution to such a small contraction is to replace X by a birationally equivalent variety with singularities milder than those of Y .

Definition 5.4 (see [KMM]) *Let $\pi : X \rightarrow Y$ be a small contraction such that $-K_X$ is π -ample. A variety X^+ together with a proper birational morphism $\pi^+ : X^+ \rightarrow Y$ is called a flip of π if π^+ is also a small contraction and K_{X^+} is π -ample.*

$$\begin{array}{ccc} X & \xrightarrow{(\pi^+)^{-1} \circ \pi} & X^+ \\ \pi \searrow & & \swarrow \pi^+ \\ & Y & \end{array} \quad (5.1)$$

Here X^+ is again \mathbf{Q} -factorial and has at worst log terminal singularities.

- If $H + T_0 K_X$ is not big, then the Kodaira dimension $0 \leq \kappa = \text{kod}(H + T_0 K_X) < n$ and X is a Mori fibre space admitting a Fano fibration over a normal variety Y of dimension κ . In particular, Y is \mathbf{Q} -factorial and has log terminal singularities.

We will discuss in the following sections the behavior of the Kähler-Ricci flow at the singular time T_0 according to the above situations.

5.2 Estimates

In this section, we assume that $T_0 < \infty$ and $H + T_0 K_X$ is big.

Let Ω be a smooth volume form on X and $\chi = \sqrt{-1}\partial\bar{\partial}\log\Omega \in [K_X]$. Consider the Monge-Ampère flow associated to the unnormalized Kähler-Ricci flow on X with the initial Kähler form ω_0 ,

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}, & [0, T_0) \times X \\ \varphi(0, \cdot) = \varphi_0 \in PSH_p(X, \omega_0, \Omega) \end{cases} \quad (5.2)$$

where $p > 1$, $\omega_t = \omega_0 + t\chi$ and $\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi$.

Since $H + T_0K_X$ is big and semi-ample, the linear system $|m(H + T_0K_X)|$ for sufficiently large m induces a morphism

$$\pi : X \rightarrow Y \subset \mathbf{CP}^{N_m}.$$

Let ω_Y be the pullback of a multiple of the Fubini-Study metric form on \mathbf{CP}^{N_m} with $\omega_Y \in [H + T_0K_X]$. There exists a resolution of singularities and the exceptional locus of π

$$\mu : \tilde{X} \rightarrow Y$$

satisfying the following conditions.

1. \tilde{X} is smooth.
2. There exists an effective divisor E_Y on \tilde{X} such that $\mu^*[H + T_0K_X] - \epsilon[E_Y]$ is ample for any sufficiently small $\epsilon > 0$ and the support of E_Y coincides with the exceptional locus of μ .

Let S_{E_Y} be the defining section for the line bundle associated to $[E_Y]$ and h_{E_Y} the hermitian metric such that for any $\epsilon > 0$,

$$\mu^*\omega_Y - \epsilon Ric(h_{E_Y}) > 0.$$

Let $Exc(\pi)$ be the exceptional locus of π . Then we have the following uniform estimates.

Theorem 5.3 *Let $\varphi \in C^0([0, T_0) \times X_{reg}) \cap C^\infty((0, T_0) \times X_{reg})$ with $\varphi(t, \cdot) \in L^\infty(X)$ for all $t \in [0, T_0)$, be the solution solving the Monge-Ampère flow (5.2). There exists a constant $C > 0$ such that*

$$\|\varphi\|_{L^\infty([0, T_0) \times X)} \leq C. \quad (5.3)$$

Furthermore, for any $K \subset\subset X \setminus Exc(\pi)$ and $k \geq 0$, there exists $C_{K,k} > 0$ such that

$$\|\varphi\|_{C^k([0, T_0) \times K)} \leq C_{K,k}. \quad (5.4)$$

Proof We lift the Monge-Ampère flow (5.2) on \tilde{X} . The proof of the L^∞ -estimate proceeds in the same way as in the proof of Lemma 3.8 since $[\omega_t]$ is big and semi-ample for all $t \in [0, T_0]$. The C^2 -estimate on \tilde{X} follows the same argument as in Lemma 3.10, which is valid on $\tilde{X} \setminus E_Y$. Since the support of $\mu(E_Y)$ is contained in $Exc(\pi)$, the C^2 -estimate holds on $X_{reg} \setminus Exc(\pi)$. We leave the details for the readers as an exercise. \square

Theorem 5.4 *There exists $C > 0$ such that on $[0, T_0) \times X$,*

$$\frac{\omega^n}{\Omega} \leq e^{\frac{C}{t}}. \quad (5.5)$$

Proof We consider $H = t\dot{\varphi} - \varphi + \epsilon \log |S_{E_Y}|_{h_Y}^2$ on \tilde{X} . Let Δ be the Laplace operator with respect to the pullback of ω . Then H is smooth outside E_Y and

$$(\frac{\partial}{\partial t} - \Delta)H = -tr_\omega(\omega_0 - \epsilon Ric(h_{E_Y})) \leq 0.$$

As $H|_{t=0} = -\varphi_0 + \epsilon \log |S_{E_Y}|_{h_Y}^2$ is bounded from above and for each $t \in (0, T_0)$, the maximum of H can only be achieved on $\tilde{X} \setminus E_Y$, $H \leq H|_{t=0}$ is uniformly bounded above and by letting $\epsilon \rightarrow 0$, there exists $C > 0$ such that

$$t\dot{\varphi} \leq C.$$

We are done. □

Corollary 5.1 *We consider the unique solution ω for the unnormalized weak Kähler-Ricci flow on X starting with an initial current in $\mathcal{K}_{H,p}(X)$ for some $p > 1$. If $H + T_0 K_X$ is big, then $\omega(t, \cdot)$ converges to a Kähler current $\tilde{\omega}_{T_0} \in \mathcal{K}_{H+T_0 K_X, \infty}(X)$ in $C^\infty(X_{reg} \setminus \text{Exc}(\pi))$ -topology. That is, there exists $C > 0$ such that*

$$\tilde{\omega}_{T_0}^n \leq C\Omega \tag{5.6}$$

for a fixed smooth volume form Ω on X .

By Corollary 5.1, H is trivial over the fibres and so ω_Y is trivial restricted on each fibre. Let $\tilde{\omega}_{T_0} = \omega_Y + \sqrt{-1}\partial\bar{\partial}\varphi_{T_0}$, then φ_{T_0} must be constant on each fibre as any fibre of π are connected. Therefore φ_{T_0} can descend onto Y and $\varphi_{T_0} \in PSH(Y, \omega_Y) \cap L^\infty(Y)$. So the limiting Kähler current $\tilde{\omega}_{T_0}$ descends onto Y as a semi-positive closed $(1, 1)$ -current.

5.3 Extending Kähler-Ricci flow through singularities by divisorial contractions

In this section, we will prove that the weak Kähler-Ricci flow can be continued through divisorial contractions.

We assume that $\pi : X \rightarrow Y$ is a divisorial contraction and the fibres of π are connected. It is well-known that Y is again a \mathbf{Q} -factorial projective variety with at worst log terminal singularities if X is.

Proposition 5.1 *Let Ω_Y be a smooth volume form on Y and $H_Y = \pi_*(H + T_0 K_X)$. Then for some $p > 1$,*

$$(\pi^{-1})^* \tilde{\omega}_{T_0} \in \mathcal{K}_{H_Y, p}(Y). \tag{5.7}$$

Proof Obviously, $\tilde{\omega}_{T_0}$ has bounded local potentia and the restriction of $\tilde{\omega}$ has constant local potential along each fibre of π . So $\tilde{\omega}_{T_0}$ descends to Y and $(\pi^{-1})^* \tilde{\omega}_{T_0}$ is well-defined and admits bounded local potential on Y . Let $F = \frac{(\tilde{\omega}_{T_0})^n}{\Omega_Y}$. It suffices to show $F \in L^p(Y, \Omega_Y)$ for some $p > 1$. There exists $C > 0$ such that

$$\int_Y F^p \Omega_Y = \int_Y \left(\frac{\tilde{\omega}_{T_0}^n}{\Omega_Y} \right)^{p-1} \tilde{\omega}_{T_0}^n = \int_X \left(\frac{\tilde{\omega}_{T_0}^n}{\pi^* \Omega_Y} \right)^{p-1} \tilde{\omega}_{T_0}^n \leq C \int_X \left(\frac{\Omega}{\Omega_Y} \right)^{p-1} \Omega.$$

Since $\frac{\Omega}{\Omega_Y}$ has at worst poles, $\int_Y F^p \Omega_Y < \infty$ for $p-1 > 0$ sufficiently small. □

Theorem 5.5 *Let X be a \mathbf{Q} -factorial projective variety with log terminal singularities and H be an ample \mathbf{Q} -divisor on X . Let*

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}$$

be the first singular time. Suppose that the semi-ample divisor $H + T_0K_X$ induces a divisorial contraction $\pi : X \rightarrow Y$.

*Let ω be the unique solution of the unnormalized weak Kähler-Ricci flow for $t \in [0, T_0)$ starting with $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$. Then there exists $\omega_{Y,0} \in \mathcal{K}_{\pi^*H_Y,p'}(Y) \cap C^\infty(Y_{\text{reg}} \setminus \pi(\text{Exc}(\pi)))$ for some $p' > 1$ such that $\omega(t, \cdot)$ converges to $\pi^*\omega_{Y,0}$ in $C^\infty(X_{\text{reg}} \setminus \text{Exc}(\pi))$ -topology as $t \rightarrow T_0$.*

Furthermore, the unnormalized weak Kähler-Ricci flow can be continued on Y with the initial Kähler current $\omega_{Y,0}$.

Proof Since H_Y is the strict transformation of H by π and $\omega_{Y,0}$ admits bounded local potential, $\omega_{Y,0} \in \mathcal{K}_{H_Y,p'}(Y)$ for some $p' > 1$ by Proposition 5.1. Then the Kähler-Ricci flow can start with $\omega_{Y,0}$ on Y uniquely as H_Y is ample. □

5.4 Extending Kähler-Ricci flow through singularities by flips

In this section, we will prove that the weak Kähler-Ricci can be continued through flips.

We assume that $\pi : X \rightarrow Y$ is a small contraction and there exists a flip

$$\check{\pi} = \pi^+ \circ \pi^{-1} : X^+ \dashrightarrow X.$$

Then X^+ is \mathbf{Q} -factorial and it has at worst log terminal singularities. The limiting Kähler current $\tilde{\omega}_{T_0}$ descends on Y and it can be then pulled back on X^+ by π^+ . Furthermore, there exists $C > 0$ such that

$$\frac{(\check{\pi}^*\tilde{\omega}_{T_0})^n}{\check{\pi}^*\Omega} \leq C. \quad (5.8)$$

Proposition 5.2 *Let Ω_{X^+} be a smooth volume form on X^+ and H_{X^+} be the strict transformation of $H + T_0K_X$ by $\check{\pi}$. Then for some $p > 1$,*

$$\check{\pi}^*\tilde{\omega}_{T_0} \in \mathcal{K}_{H_{X^+},p}(X^+). \quad (5.9)$$

Proof $(\pi^{-1})^*\tilde{\omega}_{T_0}$ is well-defined on Y with bounded local potential. Then $\check{\pi}^*\tilde{\omega}_{T_0} = ((\pi \circ \pi^+)^{-1})^*\tilde{\omega}_{T_0}$ is semi-positive closed $(1,1)$ current on X^+ with bounded local potential as well. Let $F = \frac{(\check{\pi}^*\tilde{\omega}_{T_0})^n}{\Omega_{X^+}}$. It suffices to show $F \in L^p(X^+, \Omega_{X^+})$ for some $p > 1$. There exists $C > 0$ such that

$$\begin{aligned} \int_{X^+} F^p \Omega_{X^+} &= \int_{X^+} \left(\frac{(\check{\pi}^*\tilde{\omega}_{T_0})^n}{\Omega_{X^+}} \right)^{p-1} (\check{\pi}^*\tilde{\omega}_{T_0})^n \\ &= \int_X \left(\frac{\tilde{\omega}_{T_0}^n}{(\check{\pi}^{-1})^*\Omega_{X^+}} \right)^{p-1} \tilde{\omega}_{T_0}^n \\ &\leq C \int_X \left(\frac{\Omega}{(\check{\pi}^{-1})^*\Omega_{X^+}} \right)^{p-1} \Omega. \end{aligned}$$

Since $\frac{\Omega}{(\check{\pi}^{-1})^*\Omega_{X^+}}$ has at worst poles, $\int_{X^+} F^p \Omega_{X^+} < \infty$ for $p-1 > 0$ sufficiently small. □

Theorem 5.6 *Let X be a \mathbf{Q} -factorial projective variety with log terminal singularities and H be an ample \mathbf{Q} -divisor on X . Let*

$$T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\}$$

be the first singular time. Suppose that the semi-ample divisor $H + T_0K_X$ induces a small contraction $\pi : X \rightarrow Y$ and there exists a flip

$$\begin{array}{ccc} X & \xrightarrow{\check{\pi}^{-1}} & X^+ \\ \pi \searrow & & \swarrow \pi^+ \\ & Y & \end{array} \quad (5.10)$$

Let ω be the unique solution of the unnormalized weak Kähler-Ricci flow for $t \in [0, T_0)$ starting with $\omega_0 \in \mathcal{K}_{H,p}(X)$ for some $p > 1$. Then there exists $\omega_{X^+,0} \in \mathcal{K}_{H_{X^+},p'}(X^+)$ such that $\omega(t, \cdot)$ converges to $(\check{\pi}^{-1})^ \omega_{X^+,0}$ in $C^\infty(X_{\text{reg}} \setminus \text{Exc}(\pi))$ -topology, where H_{X^+} is the strict transformation of H by $\check{\pi}$.*

Furthermore, $\omega_{X^+,0}$ is smooth outside the singularities of X^+ and where the flip is performed, and the unnormalized weak Kähler-Ricci flow can be continued on X^+ with the initial Kähler current $\omega_{X^+,0}$.

Proof Since H^+ is the strict transformation of H by $\check{\pi}$ and $\omega_{X^+,0}$ admits bounded local potential, $\omega_{X^+,0} \in \mathcal{K}_{H^+,p'}$ for some $p > 1$ by Proposition 5.2. Then the Kähler-Ricci flow can start with $\omega_{X^+,0}$ on X^+ uniquely as H^+ is big and semi-ample, and $H^+ + \epsilon K_{X^+}$ is ample for sufficiently small $\epsilon > 0$. □

5.5 Long time existence assuming MMP

As proved in Section 5.3 and 5.4, the Kähler-Ricci flow can flow through divisorial contractions and flips. If the exceptional loci of the contracted extremal rays do not meet each other, Theorem 5.5 and 5.6 still hold. However, at the singular time T_0 , the morphism $\pi : X \rightarrow Y$ induced by the semi-ample divisor $H + T_0K_X$ might contract more than one extremal ray. It simplifies the analysis to assume the existence of a good initial divisor as in Definition 5.3 as to avoid complicated contractions.

Theorem 5.7 *Let X be a \mathbf{Q} -factorial projective variety with log terminal singularities. If there exists a good initial divisor H on X , then either X does not admit a minimal model or the unnormalized weak Kähler-Ricci flow has long time existence for any Kähler current $\omega_0 \in \mathcal{K}_{H,p}(X)$ with $p > 1$, after finitely many surgeries through divisorial contractions and flips.*

Proof Assume X admits a minimal model and let $X_0 = X$ and $H_0 = H$. Since H is a good initial divisor, by MMP with scaling, at each singular time, the morphism induced by the semi-ample divisor is always a contractional contraction or flipping contraction.

More precisely, suppose the Kähler-Ricci flow performs surgeries and replaces (X_0, H_0) by a finite sequence of (X_i, H_i) at each singular time T_i , $i = 1, \dots, m$, and X_{m+1} is a minimal model of X . If λ_i is the nef threshold for (X_i, H_i) as in Definition 5.2, $i = 1, \dots, m$, $\lambda_i > 0$ and

$$T_i = T_{i-1} + \frac{1}{\lambda_i}.$$

At T_i , the morphism induced by the semi-ample divisor $H_i + T_i K_{X_i}$ contracts exactly one extremal ray and so it must be a divisorial contraction or a flip. By Theorem 5.5 and Theorem 5.6, the Kähler-Ricci flow with the pair (X_i, H_i) is replaced by the one with the pair (X_{i+1}, H_{i+1}) with H_{i+1} being the strict transform of $H_i + T_i K_{X_i}$ until X is finally replaced by its minimal model X_{m+1} and the Kähler-Ricci flow exists for all time afterwards by Theorem 4.4 as $K_{X_{m+1}}$ is nef. \square

If H is not a good initial divisor, the surgery at the finite singular time could be complicated and a detailed speculation is given in Section 6.2.

5.6 Convergence on projective varieties of general type

Let X be a minimal model of general type with log terminal singularities and so K_X is big and nef. Let H be an ample \mathbb{Q} -divisor on X and $\omega_0 \in \mathcal{K}_{H,p}$. We consider the normalized Kähler-Ricci flow on X .

$$\left\{ \begin{array}{l} \frac{\partial \omega}{\partial t} = -Ric(\omega) - \omega \\ \omega|_{t=0} = \omega_0. \end{array} \right. \quad (5.11)$$

Let Ω be a smooth volume form on X and $\chi \in \sqrt{-1}\partial\bar{\partial}\log\Omega \in c_1(K_X)$. Then the Kähler-Ricci flow (5.11) is equivalent to the following Monge-Ampère flow.

$$\left\{ \begin{array}{l} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} - \varphi \\ \varphi|_{t=0} = \varphi_0, \end{array} \right. \quad (5.12)$$

where $\omega_t = e^{-t}\omega_0 + (1 - e^{-t})\chi$.

Now that K_X is semi-ample as X is a minimal model, the abundance conjecture holds for general type. The linear system $|mK_X|$ for sufficiently large $m > 0$ induces a morphism

$$\pi : X \rightarrow X_{can},$$

where X_{can} is the canonical model of X . Without loss of generality, we can always assume that $\chi \geq 0$ and χ is big. Furthermore, we can assume $\omega_0 \geq \epsilon\chi$ for sufficiently small $\epsilon > 0$ since H is ample.

The long time existence is guaranteed by Theorem 4.4 since K_X is nef and $T_0 = \infty$.

Proposition 5.3 *The weak normalized Kähler-Ricci flow (5.11) exists on $[0, \infty) \times X$ for any initial Kähler current $\omega \in \mathcal{K}_{H,p}(X)$ with $p > 1$.*

Lemma 5.1 *There exists $C > 0$ such that*

$$\|\varphi(t, \cdot)\|_{L^\infty([0, \infty) \times X)} \leq C. \quad (5.13)$$

Proof Let \tilde{X} be a nonsingular model of X . Without loss of generality, we can consider the Monge-Ampère flow (5.12) on \tilde{X} by pullback and the following smooth approximation for the Monge-Ampère flow as discussed in Section 3.2.

$$\begin{cases} \frac{\partial \varphi_{s,w,r}}{\partial t} = \log \frac{(\omega_{t,s} + \sqrt{-1}\partial\bar{\partial}\varphi_{s,w,r})^n}{\Omega_{w,r}} - \varphi_{s,w,r} \\ \varphi_{s,w,r}|_{t=0} = \varphi_{(0,s)}, \end{cases} \quad (5.14)$$

where $\omega_{t,s} = \omega_t + s\vartheta$ and $\Omega_{w,r}$ are defined as in Section 3.2.

Let $\phi_{s,w,r} \in C^\infty(\tilde{X})$ be the solution of the following Monge-Ampère equation

$$(\chi + s\vartheta + \sqrt{-1}\partial\bar{\partial}\phi_{s,w,r})^n = e^{\phi_{s,w,r}}\Omega_{w,r}. \quad (5.15)$$

There exists $C > 0$ such that for all $s, w, r \in (0, 1]$,

$$\|\phi_{s,w,r}\|_{L^\infty(\tilde{X})} \leq C.$$

Let $\psi_\epsilon = \varphi_{s,w,r} - \phi_{s,w,r} - \epsilon \log |S_{\tilde{E}}|_{h_{\tilde{E}}}$, where \tilde{E} is a divisor whose support contains the exceptional locus of the resolution of \tilde{X} over X and $\chi - \epsilon Ric(h_{\tilde{E}}) > 0$ for sufficiently small $\epsilon > 0$. Then similar argument by the maximum principle as in Section 3.3 shows that ψ_ϵ is uniformly bounded from below for all $t \in [0, \infty)$ and for all sufficiently small $\epsilon > 0$. Then by letting $\epsilon \rightarrow 0$, there exists $C > 0$ such that for $t \in [0, \infty)$, s, w and $r \in (0, 1]$,

$$\varphi_{s,w,r} \geq -C.$$

Therefore φ is uniformly bounded from below for all $t \in [0, \infty)$ by its definition. The uniform upper bound of φ can be obtained by similar argument. \square

Lemma 5.2 *Let $X^\circ = X_{reg} \setminus Exc(\pi)$. For any $K \subset\subset X^\circ$, t_0 and $k > 0$, there exists $C_{K,k,t_0} > 0$ such that for $t \in [t_0, \infty)$,*

$$\|\varphi(t, \cdot)\|_{C_{\omega_0}^k(X)} \leq C_{K,k,t_0}. \quad (5.16)$$

Proof We can assume that X is nonsingular with the cost of ω_0 and Ω being degenerate. We first have to show that $tr_{\omega_0}(\omega)$ is uniformly bounded on K . This is achieved by similar arguments for Lemma 3.10. Then the higher order estimates follow by standard argument. \square

Lemma 5.3 *For any $t_0 > 0$, there exists $C > 0$ such that on $[t_0, \infty) \times X$,*

$$\frac{\partial \varphi}{\partial t} \leq Cte^{-t}. \quad (5.17)$$

Proof Notice that

$$(\frac{\partial}{\partial t} - \Delta) \frac{\partial \varphi}{\partial t} = -e^{-t} tr_\omega(\omega_0 - \chi) - \frac{\partial \varphi}{\partial t}.$$

Let $H = e^t \frac{\partial \varphi}{\partial t} - A\varphi + \epsilon \log |S_{\tilde{E}}|_{h_{\tilde{E}}}^2 - Ant$, where $A > 0$ is sufficiently large such that $A\omega_0 \geq \chi$ and $\epsilon > 0$ is chosen to be sufficiently small. Then there exists $C > 0$ for all sufficiently small $\epsilon > 0$ such that

$$\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)H &= -tr_\omega(A\omega_t + \omega_0 - \chi - \epsilon Rich(h_{\tilde{E}})) - A\frac{\partial\varphi}{\partial t} \\
&\leq -A\frac{\partial\varphi}{\partial t} \\
&\leq -Ae^{-t}H - A^2\varphi + A\epsilon \log |S_{\tilde{E}}|^2_{h_{\tilde{E}}} - Ant \\
&\leq -Ae^{-t}H - C.
\end{aligned}$$

Since the maximum can only be achieved on X° and $H|_{t_0=0}$ is bounded from above, by the maximum principle, there exists $C > 0$ such that on $[t_0, \infty) \times X$,

$$H \leq C(t+1).$$

Therefore there exists $C > 0$ independent of ϵ such that on $[t_0, \infty) \times X$,

$$\frac{\partial\varphi}{\partial t} \leq Ce^{-t}(1+t - \epsilon \log |S_{\tilde{E}}|^2_{h_{\tilde{E}}}).$$

The lemma is proved by letting $\epsilon \rightarrow 0$. □

Corollary 5.2

$$\lim_{t \rightarrow \infty} \|\frac{\partial\varphi}{\partial t}\|_{L^1(X)} = 0. \quad (5.18)$$

Proof There exists $T > 0$, such that $\frac{\partial\varphi}{\partial t} \leq e^{-t/2}$ for all $t \geq T$. Notice that $\int_{t_1}^{t_2} \frac{\partial\varphi}{\partial t}(t, z) dt = \varphi(t_2, z) - \varphi(t_1, z)$ is uniformly bounded for all $t_1, t_2 \geq 0$ and $\frac{\partial\varphi}{\partial t} - e^{-t/2} \leq 0$ for all $t \geq T$. Then

$$\int_T^\infty \|\frac{\partial\varphi}{\partial t}\|_{L^1(X)} dt < \infty. \quad (5.19)$$

On the other hand, $\frac{\partial}{\partial t}(\varphi + e^{-t/2}) \leq 0$ and so $\varphi + e^{-t/2}$ is decreasing in time. Since φ is uniformly bounded, there exists $\varphi_\infty \in PSH(X, \chi) \cap L^\infty(X)$ such that φ converges to φ_∞ in $L^1(X)$ and $C^\infty(X^\circ)$ as $t \rightarrow \infty$. Hence $\frac{\partial\varphi}{\partial t}$ converges to a function $F \in L^\infty(X) \cap C^\infty(X^\circ)$ in $C^\infty(X^\circ) \cap L^1(X)$. Combined with (5.19), $F = 0$. Otherwise, $\int_T^\infty \|\frac{\partial\varphi}{\partial t}\|_{L^1(X)} dt = \infty$. The corollary is then proved. □

Proposition 5.4 *Let $\varphi_\infty \in PSH(X, \chi) \cap L^\infty(X)$ be the unique solution of the Monge-Ampère equation*

$$(\chi + \sqrt{-1}\partial\bar{\partial}\varphi_\infty)^n = e^{\varphi_\infty}\Omega. \quad (5.20)$$

Then φ converges to φ_∞ in $L^1(X) \cap C^\infty(X^\circ)$ as $t \rightarrow \infty$.

Proof Let φ_∞ be the limit of φ as $t \rightarrow \infty$. By Corollary 5.2, $\frac{\partial\varphi}{\partial t}$ converges to 0, and so φ_∞ must satisfy equation (5.20). The uniqueness of φ_∞ follows from the uniqueness of the solution to the equation (5.20) as $\varphi_\infty \in PSH(X, \chi) \cap L^\infty(X)$. □

The Kähler current $\chi + \sqrt{-1}\partial\bar{\partial}\varphi_\infty$ is exactly the pullback of the unique Kähler-Einstein metric ω_{KE} on the canonical model X_{can} of X in Theorem 2.2. The following theorem then follows from Proposition 5.4.

Theorem 5.8 *Let X be a minimal model of general type with log terminal singularities. For any \mathbf{Q} -ample divisor H on X , the normalized weak Kähler-Ricci flow converges to the unique Kähler-Einstein metric ω_{KE} on the canonical model X_{can} for any initial Kähler current in $\mathcal{K}_{H,p}(X)$ with $p > 1$.*

We have the following general theorem by combining Theorem 5.7 and Theorem 5.8 if the general type variety is not minimal.

Theorem 5.9 *Let X be a projective \mathbf{Q} -factorial variety of general type with log terminal singularities. If there exists a good initial divisor H on X , then the normalized weak Kähler-Ricci flow starting with any initial Kähler current in $\mathcal{K}_{H,p}(X)$ with $p > 1$ exists for $t \in [0, \infty)$ and replaces X by its minimal model X_{min} after finitely many surgeries. Furthermore, the normalized Kähler-Ricci flow converges to the unique Kähler-Einstein metric ω_{KE} on its canonical model X_{can} .*

6 Analytic Minimal Model Program with Ricci Flow

In this section, we lay out the program relating the Kähler-Ricci flow and the classification of projective varieties following [SoT1] and [T3]. The new insight is that the Ricci flow is very likely to deform a given projective variety to its minimal model and eventually to its canonical model coupled with a canonical metric of Einstein type, in the sense of Gromov-Hausdorff. We will start discussions with the case of projective surfaces.

6.1 Results on surfaces

A smooth projective surface is minimal if it does not contain any (-1) -curve. Let X_0 be a projective surface of non-negative Kodaira dimension. If X_0 is not minimal, then the unnormalized Kähler-Ricci flow starting with any Kähler metric ω_0 in the class of an ample divisor H_0 has a smooth solution until the first singular time $T_0 = \sup\{t > 0 \mid H_0 + T_0 K_{X_0} \text{ is nef}\}$. The limiting semi-ample divisor $H_0 + T_0 K_{X_0}$ induces a morphism

$$\pi_0 : X_0 \rightarrow X_1$$

by contracting finitely many (-1) -curves. X_1 is smooth and there exists an \mathbf{Q} -ample divisor H_1 on X_1 such that $H_0 + T_0 K_{X_0} = \pi_0^* H_1$. Then by Theorem 5.5, the unnormalized Kähler-Ricci flow can be continued through the contraction π_0 at time T_0 . Since there are finitely many (-1) -curves on X , the unnormalized Kähler-Ricci flow will arrive at a minimal surface X_{min} or it collapses a \mathbf{CP}^1 fibration in finite time after repeating the same surgery for finitely many times.

It is still a largely open question if the Kähler-Ricci flow converges to the new surface in the sense of Gromov-Hausdorff at each surgery. The only confirmed case is the Kähler-Ricci flow on \mathbf{CP}^2 blow-up at one point. More precisely, it is shown in [SW] that the unnormalized Kähler-Ricci flow on \mathbf{CP}^2 blow-up at one point converges to \mathbf{CP}^2 in the sense of Gromov-Hausdorff if the initial Kähler class is appropriately chosen and the initial Kähler metric satisfies the Calabi symmetry. Then the flow can be continued on \mathbf{CP}^2 and eventually will be contracted to a point in finite time. This shows that the Kähler-Ricci flow deforms the non-minimal surface to a minimal surface in the sense of Gromov-Hausdorff. Similar behavior is also shown in [SW] for higher-dimensional analogues of the Hirzebruch surfaces. This leads us to propose a conjectural program in the following section for general projective varieties.

After getting rid of all (-1) -curves, we can focus on the minimal surfaces divided into ten classes by the Enriques-Kodaira classification.

If $\text{kod}(X_{\min}) = 2$, X_{\min} is a minimal surface of general type and its canonical model X_{can} is an orbifold surface achieved by contracting all the (-2) -curves on X_{\min} . It is shown in [TiZha] that the normalized Kähler-Ricci flow $\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) - \omega$ converges in the sense of distributions to the pullback of the orbifold Kähler-Einstein metric on the canonical model X_{can} .

If $\text{kod}(X_{\min}) = 1$, X_{\min} is a minimal elliptic fibration over its canonical model X_{can} . It is shown in [SoT1] that the normalized Kähler-Ricci flow $\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) - \omega$ converges in the sense of distributions to the pullback of the generalized Kähler-Einstein metric on the canonical model X_{can} .

If $\text{kod}(X_{\min}) = 0$, K_X is numerically trivial. Yau's solution to the Calabi conjecture shows that there always exists a Ricci-flat Kähler metric in any given Kähler class on X_{\min} . In particular, it is shown in [C] that the unnormalized Kähler-Ricci converges in the sense of distributions to the unique Ricci-flat metric in the initial Kähler class.

If X is Fano, then it is proved in [Pe2] and [TiZhu] that the normalized Kähler-Ricci flow $\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \omega$ with an appropriate initial Kähler metric will converge in the sense of Gromov-Hausdorff to a Kähler-Ricci soliton after normalization.

In general, the understanding of the Kähler-Ricci flow is still not completely understood for surfaces of $-\infty$ Kodaira dimension as the flow might collapse in finite time.

6.2 Conjectures

In this section, we discuss our program in higher dimensions. Our proposal gives new understanding of the Minimal Model Program from the viewpoint of differential geometry. We refer it as the analytic Minimal Model Program.

Minimal Model Program with Ricci Flow

1. We start with a triple (X, H, ω) , where X is a \mathbf{Q} -factorial projective variety with log terminal singularities, H is a big semi-ample \mathbf{Q} -divisor on X such that $H + \epsilon K_X$ is ample for sufficiently small $\epsilon > 0$, and $\omega \in \mathcal{K}_{H,p}(X)$ for some $p > 1$. Let

$$T_0 = \inf\{t > 0 \mid H + tK_X \text{ is nef}\}.$$

Let $\omega(t, \cdot)$ be the unique solution of the unnormalized weak Kähler-Ricci flow for $t \in [0, T_0)$.

Conjecture 6.1 *For each $t \in (0, T_0)$, the metric completion of X_{reg} by $\omega(t, \cdot)$ is homeomorphic to X .*

We also conjecture that the Ricci curvature of $\omega(t, \cdot)$ is bounded from below.

2. If $T_0 = \infty$, then X is a minimal model and the Kähler-Ricci flow has long time existence. The abundance conjecture predicts that K_X is semi-ample and $\text{kod}(X) \geq 0$.

- 2.1. $\text{kod}(X) = \dim X$, i.e., X is a minimal model of general type.

Conjecture 6.2 *The normalized Kähler-Ricci flow*

$$\frac{\partial \tilde{\omega}}{\partial s} = -\text{Ric}(\tilde{\omega}) - \tilde{\omega}.$$

starting with ω converges to the unique Kähler-Einstein metric ω_{KE} on X_{can} in the sense of Gromov-Hausdorff as $s \rightarrow \infty$.

The weak convergence in distribution and smooth convergence outside the exceptional locus is obtained in [Ts] and [TiZha] if X is nonsingular. If X is nonsingular and K_X is ample, it is the classical result in [C] that the flow converges in $C^\infty(X)$ -topology.

2.2. $0 < \text{kod}(X) < \dim X$.

Conjecture 6.3 *The normalized Kähler-Ricci flow*

$$\frac{\partial \tilde{\omega}}{\partial s} = -\text{Ric}(\tilde{\omega}) - \tilde{\omega}.$$

starting with ω converges to the unique generalized Kähler-Einstein metric ω_{can} on X_{can} (as in Theorem 2.2) in the sense of Gromov-Hausdorff as $s \rightarrow \infty$.

If K_X is semi-ample, X admits a Calabi-Yau fiberation over its canonical model X_{can} . The weak convergence in distribution is obtained in [SoT1] and [SoT2] if X is nonsingular and K_X is semi-ample.

2.3. $\text{kod}(X) = 0$. K_X is numerically trivial.

Conjecture 6.4 *The unnormalized Kähler-Ricci flow*

$$\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega)$$

converges to the unique Ricci-flat Kähler metric in [H] in the sense of Gromov-Hausdorff as $t \rightarrow \infty$.

It is shown in [C] that the flow converges in $C^\infty(X)$ -topology if X is smooth. The weak convergence is obtained in [SY] if X has log terminal singularities.

3. If $T_0 < \infty$, then the semi-ample divisor $H + T_0 K_X$ induces a contraction

$$\pi : X \rightarrow Y.$$

3.1 $\dim Y = \dim X$.

Conjecture 6.5 *As $t \rightarrow T_0$, $(X, \omega(t, \cdot))$ converges to a metric space (X^+, ω_{X^+}) along the unnormalized Kähler-Ricci flow*

$$\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega)$$

in the sense of Gromov-Hausdorff. Furthermore, $\omega(t, \cdot)$ converges in C^∞ outside a subvariety S of X , and (X^+, ω_{X^+}) is the metric completion of the smooth limit of $(X \setminus S, \omega(T_0, \cdot))$. Here X^+ is a normal projective variety satisfying the following diagram

$$\begin{array}{ccc} X & \xrightarrow{(\pi^+)^{-1} \circ \pi} & X^+ \\ & \searrow \pi & \swarrow \pi^+ & \\ & Y & & \end{array} \tag{6.1}$$

and X^+ is π^+ -ample. $\pi^+ : X^+ \rightarrow Y$ is a general flip of X .

Let H_{X^+} be the strict transformation of $H + T_0 K_X$ by the general flip. Then K_{X^+} and H_{X^+} are both \mathbf{Q} -Cartier with $H_{X^+} + \epsilon K_{X^+}$ being ample for sufficiently small $\epsilon > 0$, and $\omega_{X^+} \in \mathcal{K}_{H_{X^+}, p}(X^+)$ for some $p' > 1$.

We then repeat **Step 1** by replacing (X, H, ω) with $(X^+, H_{X^+}, \omega_{X^+})$ even though X^+ is not necessarily \mathbf{Q} -factorial. Note that a divisorial contraction is also a general flip if we choose π^+ to be the identity map.

3.2 $0 < \dim Y < \dim X$. X then admits a Fano fibration over Y .

Conjecture 6.6 *As $t \rightarrow T_0$, $(X, \omega(t, \cdot))$ converges to a metric space (Y', ω'_Y) along the unnormalized Kähler-Ricci flow*

$$\frac{\partial \omega}{\partial t} = -Ric(\omega)$$

in the sense of Gromov-Hausdorff. Let $H_{Y'}$ be the divisor where $\omega_{Y'}$ lies. Then both $K_{Y'}$ and $H_{Y'}$ are \mathbf{Q} -Cartier, and $\omega_{Y'} \in \mathcal{K}_{H_{Y'}, p'}(Y')$ for some $p' > 1$.

We then repeat **Step 1** by replacing (X, H, ω) by $(Y', H_{Y'}, \omega_{Y'})$.

3.3 If $\dim Y = 0$, X is Fano and $\omega \in -T_0[K_X]$.

Then we have the following generalized Hamilton-Tian conjecture.

Conjecture 6.7 *Then the normalized Kähler-Ricci flow*

$$\frac{\partial \tilde{\omega}}{\partial s} = -Ric(\tilde{\omega}) + \frac{1}{T_0} \tilde{\omega}.$$

starting with ω converges to a Kähler-Ricci soliton (X_∞, ω_{KR}) in the sense of Gromov-Hausdorff as $s \rightarrow \infty$.

Perelman [Pe2] announced a proof for this conjecture for Kähler-Einstein manifolds. A proof is given for Fano manifolds with a Kähler-Ricci soliton by Tian-Zhu [TiZhu].

It is conjectured by Yau [Y2] that the existence of a Kähler-Einstein metric on a Fano manifold is equivalent to suitable stability in the sense of geometric invariant theory. The condition of K -stability is proposed by Tian [T1] and is refined by Donaldson [Do]. The Yau-Tian-Donaldson conjecture claims that the existence of Kähler metrics with constant scalar curvature is equivalent to the K -stability (possibly with some additional milder conditions on holomorphic vector fields). Since the Kähler-Ricci flow provides an approach to such a conjecture for Kähler-Einstein metrics and it has attracted considerable current interest. We refer the readers to an incomplete list of literatures [PS1], [PS2], [TiZhu], [PSSW1], [PSSW2], [Sz] and [To] for some recent development.

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